

Utilization of Bundle Space formulation in classification of physical fields.

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Abstract:

In this paper we discuss the classification of bundles for physical fields, and illustrate the importance of classification problem for these physical

Fields. The geometrical description will be global and free of coordinates.

Key words: Principal bundle, aspherical, Eilenberg-MacLane space, Simplicial set, geometric realization.

المستخلص :

في هذه الورقة ناقشنا تصنيف الحزم لأجل الحقول الفيزيائية وأوضحنا أهمية مسألة

التصنيف لأجل الحقول الفيزيائية. الوصف الهندسي شامل ويعطي وصف إحدائي حر.

Introduction:

We show that universal bundle exist for every group G , hence there is a classifying space BG which is unique up to homotopy type so that the homotopy classes of maps from a space X , $[X, BG]$, is in bijective correspondence with the set of

isomorphism classes of principal G - bundle $prin_G(X)$. We describe example and two constructions of universal bundle one due to Milnor that involves taking the infinite join of a group with itself. The other is an example of a simplicial space called

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the simplicial bar construction. These constructions are essentially equivalent.

Universal bundle and classifying space:

Definition (1) Let G be a topological group, A principal G – bundle is a fiber bundle $p : E \rightarrow B$ with fiber $F = G$ satisfying the following properties :

(1) The total space E has a free right G action. That is, it has a free group action making the following diagram commute :

$$\begin{array}{ccc} E \times G & \xrightarrow{\mu} & E \\ \varepsilon \downarrow & & \downarrow p \\ B \times \{1\} & = & B \end{array}$$

Where ε is the constant map.

(2) The induced action on fiber $p^{-1}(x) \times G \rightarrow p^{-1}(x)$ is free and transitive (3) There exists equivariant. That is, the following diagrams commute :

$$\begin{array}{ccc} & \psi & \\ P^{-1}(U) \times G & \xrightarrow{\mu} & U \times G \times G \\ & \cong & \\ \mu \downarrow & & \downarrow 1 \times \text{mult} \\ & \cong & \\ P^{-1}(U) & \xrightarrow{\psi} & U \times G \\ & \psi & \end{array}$$

Theorem (1) : Covering Homotopy Theorem. Let $p : E \rightarrow B$ and $q : Z \rightarrow Y$ be fiber bundles with the same fiber F , where B is normal and locally compact. let h_0 be a bundle map

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_0} & Z \\ p \downarrow & & \downarrow q \\ B & \xrightarrow{h_0} & Y \end{array}$$

Let $H : B \times I \rightarrow Y$ be a homotopy of h_0 (i.e $h_0 = H|_{B \times \{1\}}$). Then there exist a covering of the homotopy H by a bundle map

$$\begin{array}{ccc}
 E \times I & \xrightarrow{\tilde{H}} & Z \\
 p \times 1 \downarrow & & \downarrow q \\
 B & \xrightarrow{H} & Y
 \end{array}$$

Theorem (2): Let $p : E \rightarrow B$ be a fiber bundle with fiber F , and let $f_0 : X \rightarrow B$ and $f_1 : X \rightarrow B$ be homotopic maps. Then the pull - back bundles are isomorphic $f_0^*(E) \cong f_1^*(E)$.

Proof :_since $H : B \times I \rightarrow Y$ is a homotopy with $H_0 = f_0$ and $H_1 = f_1$. by a homotopy covering theorem there is a covering homotopy $\tilde{H} : f_0^*(E) \times I \rightarrow E$ that covers $H : X \times I \rightarrow B$. This

$$\begin{array}{ccc}
 f_0^*(E) \times I & \xrightarrow{\tilde{H}} & f_0^*(E) \\
 \downarrow & & \downarrow \\
 X \times I & \xrightarrow{=} & X \times I
 \end{array}$$

defines a map of bundles over $X \times I$

and this is a bundle isomorphism . 2^e it induces the identity map on both fibers. Restricting this isomorphism to $X \times \{1\}$, since $H_1 = f_1$, we get a bundle isomorphism

$$\begin{array}{ccc}
 f_0^*(E) & \xrightarrow{\tilde{H}} & f_1^*(E) \\
 \cong & & \cong \\
 \downarrow & & \downarrow \\
 X \times I & \xrightarrow{=} & X \times I
 \end{array}$$

Corollary (1) : Let $p : E \rightarrow B$ be a principal G - bundle over a connected space B . Then for any space X the pull back construction gives a well defined map from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G - bundles

$$\rho_E : [X, B] \rightarrow \text{prin}_G(X)$$

Definition (2) : A principal G – bundle $p : EG \rightarrow BG$ is called universal if the pull back construction $\rho_{EG} : [X, B] \rightarrow \text{prin}_G(X)$ is a bijection for every space X . In this case the base space of universal bundle BG is called a classifying space for G (or a principal G – bundle).

Corollary (2) : If $f : X \rightarrow Y$ is a homotopy equivalence, then

$$\text{it induces isomorphisms } f^* : \text{prin}_G(Y) \xrightarrow{\cong} \text{prin}_G(X)$$

$$\text{Vect}^*(Y) \xrightarrow{\cong} \text{Vect}^*(X)$$

Where $\text{Vect}^*(Y)$ and $\text{Vect}_R^*(Y)$ denote the set of isomorphism classes of real and complex vector bundles over Y respectively.

Theorem (3) : There is a bijective correspondence between principal bundles and homotopy groups $\text{prin}_G(S^n) \cong \Pi_{n-1}(G)$, where as a set $\Pi_{n-1}(G) = [S^{n-1}, x_0; G, \{1\}]$, which refers to homotopy classes of base point preserving maps from the sphere S^{n-1} with base point $x_0 \in S^{n-1}$, to the group G with base point the identity $1 \in G$.

Definition (4) : A space X is said to be aspherical if all of its homotopy groups are trivial, $\Pi_n(X) = 0, \forall n \geq 0$. Equivalently, a space X is aspherical if every map from a sphere $S^n \rightarrow X$ can be extended to a map of its bounding disk, $D^{n+1} \rightarrow X$.

The following results identifies when a principal bundle is universal.

Theorem (4) : Let $p : E \rightarrow B$ be a principal G – bundle, where the total space E is aspherical. Then this bundle is universal in the sense that if X is any space, the induced pull –back map

$\psi : [X, B] \rightarrow \text{prin}_G(X)$ defined by $f \mapsto f^*(E)$ is a bijective correspondence.

Theorem (5): Let $E_1 \rightarrow B_1$ and $E_2 \rightarrow B_2$ be universal principal G-bundles. Then there is a bundle map

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{h}} & E_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{h} & B_2 \end{array}$$

So that h is a homotopy equivalence.

Proof : since $E_2 \rightarrow B_2$ is a universal bundle, by (4) there is a classifying map $f: B_1 \rightarrow B_2$ and isomorphism $\tilde{f}: E_1 \rightarrow h^*(E_2)$, \tilde{h} can be thought of as a bundle map $\tilde{f}: E_1 \rightarrow E_2$ lying over h. Similarly we get a classifying map $g: B_2 \rightarrow B_1$ and isomorphism $\tilde{g}: E_2 \rightarrow g^*(E_1)$, or equivalently, a bundle map $\tilde{g}: E_2 \rightarrow E_1$, the composition $g \circ f: B_1 \rightarrow B_2 \rightarrow B_1$ is a map

$$\begin{aligned} (g \circ f)^*(E_1) &= g^*(f^*(E_1)) \\ &\cong g^*(E_2) \\ &\cong E_1 \end{aligned}$$

whose pull back

That is, $(g \circ f)^*(E_1) \cong id^*(E_1)$, and hence by (4) we have $g \circ f \simeq id: B_1 \rightarrow B_1$. Similarly $f \circ g \simeq id: B_2 \rightarrow B_2$. Thus f and g are homotopy inverses for each other.

Definition (3) : An Eilenberg- Maclane space of type (G, n) is the space X such that

$$\Pi_k(X) = \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

we write $K(G, n)$ for an Eilenberg- Maclane space of type (G, n) . Recall that for $n \geq 2$, the homotopy groups $\Pi_n(X)$ are abelian groups, so in this $K(G, n)$ only exists.

Corollary (3) : Let π be a discrete group. Then the classifying space $B\pi$ is an Eilenberg- Maclane space of type $K(\pi, 1)$.

Examples (1) [8] : 1- The inclusion $S^n \subset S^{n+1}$ is null homotopic. Hence the direct limit space $\lim_{n \rightarrow \infty} S^n = \bigcup_n S^n = S^\infty$ is aspherical. Z_2 acts freely on S^n .

The inclusions $S^n \subset S^{n+1}$ are equivariant and hence induce free action of Z_2 on S^∞ . The projection map $S^\infty \rightarrow Z = RP^\infty$ is a universal principal

$$Z_2 = O(1)\text{-bundle, and so } RP^\infty = BO(1) = BZ = K(Z_2, 1).$$

2-The inclusion of the unit sphere in C^n into the unit sphere in C^{n+1} gives an inclusion $S^{2n-1} \subset S^{2n+1}$ which is null homotopic. It is equivariant with respect to the free $S^1 = U(1)$ -

action. Then, the limit $\lim_{n \rightarrow \infty} S^{2n-1} = \bigcup_n S^{2n-1}$ is a spherical with

a free S^1 action. the projection $S^\infty \rightarrow S^\infty / S^1 = CP^\infty$ is a principal $S^1 = U(1)$ -bundle. Hence, $CP^\infty = BS^1 = BU(1)$. Since S^1 is $K(Z, 1)$, then $CP^\infty = K(Z, 2)$.

Theorem (6) : Let G be an abelian group. There is a natural isomorphism $\phi : H^n(K(G, n); G) \xrightarrow{\cong} Hom(G, G)$. Let $\iota \in H^n(K(G, n); G)$ be $\phi^{-1} id$. This is called the fundamental class. Then if X has the homotopy type of a CW- complex, the mapping $[X, K(G, n)] \rightarrow H^n(X; G)$ such that $f \mapsto f^*(\iota)$ is a bijective correspondence.

Theorem(7) : There are bijective correspondences which allow us to classify complex line bundles,

$$Vect^1(X) \cong prin_{U(1)}(X) \cong [X, BU(1)] = [X, CP^\infty] \cong [X, K(Z, 2)] \cong H^2(X; Z)$$

Where the last correspondence takes a map $f : X \rightarrow CP^\infty$ to the class $c_1 = f^*(c) \in H^2(X)$, where $c \in H^2(CP^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\xi \in Vect^1(X)$ is called the first chern class of ξ (or of the corresponding principal U(1)- bundle).

Theorem (8) : There are bijective correspondences

$$Vect^1_R(X) \cong prin_{O(1)}(X) \cong [X, BO(1)] = [X, RP^\infty] \cong [X, K(Z_2, 1)] \cong H^1(X; Z_2)$$

where the last correspondence takes a map $f : X \rightarrow RP^\infty$ to the class $\omega_1 = f^*(\omega) \in H^1(X; Z_2)$, wh $\omega \in H^1(RP^\infty; Z_2)$ is the generator. In the composition of the correspondences, the class $\omega_1 \in H^1(X; Z_2)$ corresponding to a line bundle $\xi \in Vect^1_R(X)$ is called the first Stiefel-Whitney class of ξ (or of the corresponding principal O(1)- bundle).

Let $A(P)$ be the space of connections. let $x_0 \in M$ be a fixed base point, and P_{x_0} be the fiber of P at x_0 .

Definition (4) : The based gauge group $\zeta_0^{(P)}$ is a subgroup of gauge group $\zeta^{(P)}$ which point wise fix the fiber P_{x_0} . That is $\zeta_0^{(P)} = \{ \phi \in \zeta^{(P)} : \text{if } v \in P_{x_0} \text{ then } \phi(v) = v \}$ let $B(P)$ and $B_0^{(P)}$ be the orbit spaces of connections on P up to gauge and based gauge equivalence respectively $B(P) = A(P) / \zeta^{(P)}$, $B_0^{(P)} = A(P) / \zeta_0^{(P)}$.

Theorem (9) : Let $G \rightarrow EG \rightarrow BG$ be a universal principal bundle for the Lie group G (so that EG is a spherical). Let

$y_0 \in BG$ be a fixed base point. There are homotopy equivalences

$$B_{\zeta}(P) \simeq \text{Map}^P(M, BG) \quad \text{and}$$

$B_0(P) = B_{\zeta_0}(P) \simeq \text{Map}_0^P(M, BG)$, where $\text{Map}(M, BG)$ is the space of all continuous maps from M to BG and $\text{Map}_0(M, BG)$ is the space of those maps that preserve the base points. The superscript P denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal G - bundle P .

Constructions of the universal bundle:

Definition (5) : The **Join** $X * Y$ is defined by $X * Y = X \times I \times Y / \square$ where $I = [0,1]$ is the unit interval and the equivalence relation is given by $(x, 0, y_1) \sim (x, 0, y_2)$ for any two points $y_1, y_2 \in Y$, and similarly $(x_1, 1, y) \sim (x_2, 1, y)$ for any two points $x_1, x_2 \in X$. A point $(x, t, y) \in X * Y$ should be viewed as a point on the line connecting the points x and y .

Lemma (1) : The inclusions $\iota: X \rightarrow X * Y$ and $j: Y \rightarrow X * Y$ are null homotopic.

Let G be a group and consider the **iterated join** $G^{*(k+1)} = G * G * \dots * G$ where there are $k+1$ copies of the group element. This space has a free G action given by the diagonal action $g \cdot (g_0, t_1, g_1, \dots, t_k, g_k) = (gg_0, t_1, gg_1, \dots, t_k, gg_k)$.

Define $\mathfrak{J}(G)$ to be the infinite join $\mathfrak{J}(G) = \lim_{k \rightarrow \infty} G^{*(k+1)}$ where the limit is taken over the embeddings $\iota: G^{*(k+1)} \rightarrow G^{*(k+2)}$ since these embedding maps are G -equivariant, we have an induced G -action on $\mathfrak{J}(G)$.

Theorem (10) : The projection map $p: \mathfrak{J}(G) \rightarrow \mathfrak{J}(G)/G$ is a universal principal G - bundle.

$\mathfrak{S}(G)$ has the structure of a G -CW-complex with a free G -action, the projection $\mathfrak{S}(G) \rightarrow \mathfrak{S}(G)/G$ is a principal G -bundle. To see that $\mathfrak{S}(G)$ is a spherical, since S^n is compact, any map $\alpha: S^n \rightarrow \mathfrak{S}(G)$ is homotopic to one that factors through a finite join, $\alpha: S^n \rightarrow G^{*(n+1)} \rightarrow \mathfrak{S}(G)$ the inclusion $G^{*(n+1)} \subset \mathfrak{S}(G)$ is null homotopic and hence α . Thus $\mathfrak{S}(G)$ is a spherical, the projection $\mathfrak{S}(G) \rightarrow \mathfrak{S}(G)/G$ is a universal G -bundle. The total space $EG = \mathfrak{S}(G)$ can be described as the union of simplices, where the k -simplices are parametrized by $k+1$ -tuples of elements of G ,

$$EG = \mathfrak{S}(G) = \bigcup_k \Delta^k \times G^{k+1} / \square$$

And the classifying space can be described by

$$BG = \mathfrak{S}(G)/G \cong \bigcup_k \Delta^k \times G^{k+1} / \square$$

In these constructions, the simplices are glued together along faces. If X_n denotes the indexing set for the n -dimensional simplices of X , then we can write $X = \bigcup_{n \geq 0} \Delta^n \times X_n / \square$

where Δ^n is the standard n -simplex in R^n .

$$\Delta^n = \{(t_1, \dots, t_n) \in R^n : 0 \leq t_j \leq 1 \text{ and } \sum_{i=1}^n t_i \leq 1\}$$

Definition (6) : A simplicial set X_* is a collection of sets X_n , $n \geq 0$ together with set maps $\partial_i: X_n \rightarrow X_{n-1}$ and $s_j: X_n \rightarrow X_{n+1}$ for $0 \leq i, j \leq n$ called face and degeneracy map respectively. These maps are required to satisfy the following compatibility conditions

$$\partial_i \partial_j = \partial_{j-1} \partial_i \quad \text{for } i < j$$

$$S_i S_j = S_{j+1} S_i \quad \text{for } i < j$$

$$\partial_i S_j = \begin{cases} S_{j-1} \partial_i & \text{for } i \leq j \\ 1 & \text{for } i = j, j+1 \\ S_j \partial_{i-1} & \text{for } i \geq j+1 \end{cases}$$

And

the maps ∂_i and S_j encode the combinatorial information necessary for gluing the simplices together. To see this, consider the following maps between the standard simplices

$\delta_i: \Delta^{n-1} \rightarrow \Delta^n$ and $\sigma_j: \Delta^{n+1} \rightarrow \Delta^n$ for $0 \leq i, j \leq n$ defined by

$$\delta_i(t_1, \dots, t_{n-1}) = \begin{cases} (t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) & i \geq 1 \\ (1 - \sum_{q=1}^n t_q, t_1, \dots, t_{n-1}) & i = 0 \end{cases}$$

$$\sigma_i(t_1, \dots, t_{n+1}) = \begin{cases} (t_1, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}) & i \geq 1 \\ (t_2, \dots, t_{n+1}) & i = 0 \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i th face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j th face.

Definition (7) : The geometric realization of a simplicial set

$$\|X_*\| = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

X_* is the space

Where if $t \in \Delta^{n-1}$ and $x \in X_n$, then $(t, \partial_i(x)) \sim (\delta_i(t), x)$.

if $t \in \Delta^{n+1}$ and $x \in X_n$, then $(t, S_j(x)) \sim (\sigma_j(t), x)$.

Let X_* be any simplicial set, we compute the homology of the geometric realization, $H_*(\|X_*\|)$. Define $C_n(X_*)$ to be the free abelian group generated by the set of n -simplices X_n . Define the homomorphism $d_n: C_n(X_*) \rightarrow C_{n-1}(X_*)$

$$d_n([x]) = \sum_{i=1}^n (-1)^i \partial_i([x]), \quad x \in X_n$$

Proposition (1) : The homology of the geometric realization $H_*(\|X_*\|)$ is the homology of the chain complex $\rightarrow \dots \xrightarrow{d_{n+1}} C_n(X_*) \xrightarrow{d_n} C_{n-1}(X_*) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_0} C_0(X_*)$.

We use simplicial theory to construct universal principal G-bundle and the classifying space. Let G be a topological group and let εG_* be the simplicial space defined as follows. The space of n- simplices is given by the n+1 - fold Cartesian product

$$\varepsilon G_n = G^{n+1} \quad \mathbf{8}$$

The face maps $\partial_i : G^{n+1} \rightarrow G^n$

$$\partial_i(g_0, \dots, g_{n+1}) = (g_0, \dots, g_j, g_j, \dots, g_{n+1})$$

The degeneracy maps $S_j : G^{n+1} \rightarrow G^{n+2}$

$$S_j(g_0, \dots, g_{n+1}) = (g_0, \dots, g_j, g_j, \dots, g_{n+1})$$

The group G acts freely on the right of

$$\|\varepsilon G_*\| \times G = \left(\bigcup_{p \geq 0} G^{p+1} \right) \times G \rightarrow \|\varepsilon G_*\|$$

$$(t; (g_0, \dots, g_p)) \times g \mapsto (t, g_0 g, \dots, g_p g)$$

$EG = \|\varepsilon G_*\|$, we define the projection map

$p : EG \rightarrow EG/G = BG$ is a universal principal G - bundle. This description gives the classifying space BG an induced simplicial structure described as follows :

Let BG_* be the simplicial space whose n- simplices are the Cartesian product $BG_n = G^n$

The face maps $\partial_i : G^n \rightarrow G^{n-1}$

$$\partial_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 1 \leq i \leq n-1 \\ (g_2, \dots, g_n) & i = n \end{cases}$$

The degeneracy maps $S_j: G^n \rightarrow G^{n+1}$

$$S_j(g_1, \dots, g_n) = \begin{cases} (1, g_1, \dots, g_n) & j = 0 \\ (g_1, \dots, g_j, g_{j+1}, \dots, g_n) & j \geq 1 \end{cases}$$

The simplicial projection map $p: \varepsilon G_* \rightarrow BG_*$ defined on the level of n- simplices by

$$p(g_0, \dots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \dots, g_{n-1} g_n^{-1})$$

So it induces a map on the level of geometric realizations

$$p: EG = \|\varepsilon G_*\| \rightarrow \|BG_*\|$$

Which induces a homeomorphism

$$BG = EG/G \xrightarrow{\cong} \|BG_*\|$$

Thus for any topological group this construction gives a simplicial space model for its classifying space. This is referred to as the simplicial bar construction since G is discrete the classifying space $BG = K(G, 1)$.

Line bundle over projective space : by the classification theorem we know that the set of isomorphism classes of complex line bundles over the projective space CP^n is given by

$$Vect^1(CP^n) \cong \text{Prin}_{GL(1, \mathbb{C})}(CP^n) \cong \text{Prin}_{U(1)}(CP^n) \cong [CP^n, BU(1)] = [CP^n, CP^\infty] = [CP^n, K(\mathbb{Z}, 2)] \cong H^2(CP^n, \mathbb{Z}) \cong \mathbb{Z}$$

Conclusion : In this paper we illustrate geometrical description for using bundle formulation to classify fields. We find a result, that is for every group there is a classifying space. We hope to study the application of fiber bundle for physical fields using characteristic classes in the next papers.

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