

## A generalization of Bounded Toeplitz products on Bergman spaces of the unit balls

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### Abstract:

In this study we consider the product  $T_f T_{\bar{g}}$  on the weighted Bergman spaces of the unit balls where  $f$  and  $g$  are square integrable analytic functions. Other results have been shown on unit disc and polydisc.

### Introduction:

The study of this problem was initiated by Sarason [7] in the context of the Hardy space  $H^2$  of the unit circle, often he had obtained of functions  $f$  and  $g$  in  $H^2$  such that the product  $T_f T_{\bar{g}}$  is bounded on  $H^2$ , while neither  $T_f$  nor  $T_g$  is bounded [5,6]. In the context of the Hardy space, the Poisson Kernel plays the role of the Berezin transform-Treil showed that a condition analogous to (6) is necessary [7], while the second author proved that a condition analogous to (7) is sufficient [8].

The above results are analogous to [10], and generalize the results in

[9, 11]. However, the proofs require new tools to establish the necessary condition, as well as consideration of higher derivatives, and an inner product formula involving higher derivatives. Recently Park [3] has proved a necessary and close-to-sufficient condition for Toeplitz product on the

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Bergman space of the ball. In addition to consideration of higher-order derivatives, this required him to find a new way to rewrite an inner product formula. Park's method to obtain an inner product suitable to prove sufficiency does not work for weighted Bergman spaces on the ball in this paper we will use a novel approach to obtain a suitable inner product formula. In extending the necessity result to the weight setting on the ball, we make use of an identity that played a key role in the argument for weighted Bergman spaces on the disk [11] as well as an estimate inspired by recent result of Arazy and Englis [1, 2].

### 1- Preliminaries:

Throughout let  $n$  be a fixed integer  $n \geq 2$ . Denote the unit ball in  $\mathbb{C}^n$  by  $\mathbb{B}_n$ , and let  $\nu$  be Lebesgue volume normalized so that  $\nu(\mathbb{B}_n) = 1$ .

for  $\omega_r \in \mathbb{B}_n$ , let  $\varphi_{\omega_r}$  be the automorphisms of  $\mathbb{B}_n$  such that  $\varphi(0) = \omega_r$  and  $\varphi_{\omega_r}^{-1} = \varphi_{\omega_r}$ . The mappings  $\varphi_{\omega_r}$  are described in [4].

for  $-1 < \alpha < \infty$ . We denote by  $\nu_\alpha$  the measure on  $\mathbb{B}_n$  defined by

$$\sum_{r=1}^n dv_\alpha(z_r) = \sum_{r=1}^n (1 - |z_r|^2)^\alpha d\nu(z_r).$$

The weighted Bergman space  $A_\alpha^2(\mathbb{B}_n)$  is the space of analytic functions  $h_r$  on  $\mathbb{B}_n$  which are square-integrable with respect to measure  $\nu_\alpha$  on  $\mathbb{B}_n$ . The reproducing Kernel in  $A_\alpha^2(\mathbb{B}_n)$  is given by

$$\sum_{r=1}^n K_{\omega_r}^{(\alpha)}(z_r) = \sum_{r=1}^n \frac{1}{(1 - (z_r, \omega_r))^{n+\alpha+1}} \tag{1}$$

for  $z, \omega_r \in \mathbb{B}_n$ . If  $\langle \cdot, \cdot \rangle_\alpha$  denote the inner product in  $L^2(\mathbb{B}_n, d\nu_\alpha)$ , then  $\sum_{r=1}^n \langle h_r, k_{\omega_r}^{(\alpha)} \rangle_\alpha = \sum_{r=1}^n h_r(\omega_r)$ , for every  $h_r \in A_\alpha^2(\mathbb{B}_n)$  and  $\omega_r \in \mathbb{B}_n$ . In this paper we use  $\|\cdot\|_\alpha$  to denote the norm in  $L^2(\mathbb{B}_n, d\nu_\alpha)$ . The orthogonal projection  $P_\alpha$  of  $L^2(\mathbb{B}_n, d\nu_\alpha)$  onto  $A_\alpha^2(\mathbb{B}_n)$  is given by

$$\begin{aligned} \sum_{r=1}^n (P_\alpha g(z_r))(\omega_r) &= \sum_{r=1}^n \langle g, k_{\omega_r}^{(\alpha)} \rangle_\alpha \\ &= \int_{\mathbb{B}_n} \sum_{r=1}^n g(z_r) \frac{1}{(1 - (z, \omega_r))^{n+\alpha+1}} d\nu_\alpha(z_r) \end{aligned} \tag{2}$$

for  $g \in L^2(\mathbb{B}_n, d\nu_\alpha)$  and  $\omega_r \in \mathbb{B}_n$ . Given  $f \in L^\infty(\mathbb{B}_n)$ , The Toeplitz operator  $T_f$  is defined on

$A_{\alpha}^2(\mathbb{B}_n)$  by  $\sum_{r=1}^n T_f h_r = \sum_{r=1}^n p_{\alpha}(f h_r)$ , we have

$$\sum_{r=1}^n (T_f h_r)(\omega_r) = \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{f(z_r) h(z_r)}{(1-\langle z_r, \omega_r \rangle)^{n+\alpha+1}} dv_{\alpha}(z_r) \quad (3)$$

for  $h_r \in A_{\alpha}^2(\mathbb{B}_n)$  and  $\omega_r \in \mathbb{B}_n$ , note that the above formula makes sense, and defines a function analytic on  $\mathbb{B}_n$ , also if  $f \in L^2(\mathbb{B}_n, dv_{\alpha})$ . So, if  $g \in A_{\alpha}^2(\mathbb{B}_n)$  we define  $T_{\bar{g}}$  as the formula

$$\sum_{r=1}^n (T_{\bar{g}} h_r)(\omega_r) = \sum_{r=1}^n \int_{\mathbb{B}_n} \frac{\overline{g(z_r)} h(z_r)}{(1-\langle z_r, \omega_r \rangle)^{n+\alpha+1}} dv_{\alpha}(z_r) \quad (4)$$

for  $h_r \in A_{\alpha}^2(\mathbb{B}_n)$  and  $\omega_r \in \mathbb{B}_n$ . If so  $f \in A_{\alpha}^2(\mathbb{B}_n)$  then  $T_f T_{\bar{g}} h_r$  is the analytic function  $f T_{\bar{g}} h_r \in H^{\infty}(\mathbb{B}_n)$  and  $\omega_r \in \mathbb{B}_n$ , will give a necessary condition for boundedness of the Toeplitz product  $T_f T_{\bar{g}}$  (in section 3) and then show that this condition is very close to being sufficient (in section 4) our conditions are formulated in terms of the (weighted) Berezin transform. For a function  $u \in L^1(\mathbb{B}_n, dv_{\alpha})$ , the Berezin transform  $\mathbb{B}_{\alpha}[u]$  is the function on  $\mathbb{B}_n$  defined by

$$\begin{aligned} & \sum_{r=1}^n \mathbb{B}_{\alpha}[u](\omega_r) \\ &= \int_{\mathbb{B}_n} \sum_{r=1}^n u(z_r) \frac{(1-|\omega_r|^2)^{n+\alpha+1}}{|1-\langle z_r, \omega_r \rangle|^{2n+2+2\alpha}} dv_{\alpha}(z_r), \omega_r \in \mathbb{B}_n \end{aligned} \quad (5)$$

We prove that a necessary condition for boundedness of the Toeplitz product  $T_f T_{\bar{g}}$  on  $A_{\alpha}^2(\mathbb{B}_n)$  is that

$$\sum_{r=1}^n \sup_{\omega_r \in \mathbb{B}_n} \mathbb{B}_{\alpha}[|f|^2](\omega_r) \mathbb{B}_{\alpha}[|g|^2](\omega_r) < \infty \quad (6)$$

We also prove that a slightly stronger condition is sufficient if for every  $\varepsilon > 0$

$$\sum_{r=1}^n \sup_{\omega_r \in \mathbb{B}_n} \mathbb{B}_{\alpha}[|f|^{2+\varepsilon}](\omega_r) \mathbb{B}_{\alpha}[|g|^{2+\varepsilon}](\omega_r) < \infty, \quad (7)$$

then the operator  $T_f T_{\bar{g}}$  is bounded on  $A_{\alpha}^2(\mathbb{B}_n)$ .

Using the reproducing property of  $\sum_{r=1}^n K_{(\omega_r)}^{(\alpha)}$ , We have

$$\begin{aligned} & \sum_{r=1}^n \left\| K_{\omega_r}^{(\alpha)} \right\|_{\alpha}^2 = \sum_{r=1}^n \langle K_{\omega_r}^{(\alpha)}, K_{\omega_r}^{(\alpha)} \rangle_{\alpha} \\ &= \sum_{r=1}^n K_{\omega_r}^{(\alpha)}(\omega_r) = \sum_{r=1}^n \frac{1}{(1-|\omega_r|^2)^{n+\alpha+1}} \end{aligned} \quad (8)$$

Thus the normalized reproducing Kernel is given by

$$\sum_{r=1}^n K_{\omega_r}^{(\alpha)}(z_r) = \sum_{r=1}^n \frac{(1-|\omega_r|^2)^{n+\alpha+1/2}}{(1-\langle z_r, \omega_r \rangle)^{n+\alpha+1}} \quad (9)$$

for  $z_r, \omega_r \in \mathbb{B}_n$ . For  $\omega_r \in \mathbb{B}_n$  the function  $\varphi_{\omega_r}$  has real Jacobian equal to

$$\sum_{r=1}^n \left| \varphi_{\omega_r}'(z_r) \right|^2 = \sum_{r=1}^n \frac{(1-|\omega_r|^2)^{n+1}}{|1-\langle z_r, \omega_r \rangle|^{n+2}} \quad (10)$$

Thus we have the change-of-variable formula

$$\begin{aligned} & \int_{B_n} \sum_{r=1}^n h_r(\varphi_{\omega_r}(z_r)) \left| K_{\omega_r}^{(\alpha)}(z_r) \right|^2 dv_{\alpha}(z_r) \\ &= \int_{B_n} \sum_{r=1}^n h_r(u) dv_{\alpha}(z_r) \end{aligned} \tag{11}$$

For every  $h_r \in L^1(\mathbb{B}_n, dv_{\alpha})$ . It follows from (11) that the mapping  $\sum_{r=1}^n U_{\omega_r}^{*(\alpha)} h_r = \sum_{r=1}^n (h_r \circ \varphi_{\omega_r}) K_{\omega_r}^{(\alpha)}$  is an isometry on  $A_{\alpha}^2(\mathbb{B}_n)$ :

$$\begin{aligned} \sum_{r=1}^n \left\| U_{\omega_r}^{*(\alpha)} h_r \right\|_{\alpha}^2 &= \int_{\mathbb{B}_n} \sum_{r=1}^n |h_r(\varphi_{\omega_r}(z_r))|^2 \left| K_{\omega_r}^{(\alpha)}(z_r) \right|^2 dv_{\alpha}(z_r) \\ &= \int_{\mathbb{B}_n} \sum_{r=1}^n |h_r(u)|^2 dv_{\alpha}(z_r) = \sum_{r=1}^n \|h_r\|_{\alpha}^2 \end{aligned} \tag{12}$$

for all  $h_r \in A_{\alpha}^2(\mathbb{B}_n)$ . Using the identity

$$\sum_{r=1}^n 1 - \langle \varphi_{\omega_r}(z_r), \omega_r \rangle = \sum_{r=1}^n \frac{1 - |\omega_r|^2}{1 - \langle z_r, \omega_r \rangle} \tag{13}$$

We have

$$\begin{aligned} \sum_{r=1}^n K_{\omega_r}^{(\alpha)}(\varphi_{\omega_r}(z_r)) &= \sum_{r=1}^n \frac{(1 - |\omega_r|^2)^{(n+\alpha+1)/2}}{(1 - \langle \varphi_{\omega_r}(z_r), \omega_r \rangle)^{n+\alpha+1}} \\ &= \sum_{r=1}^n \frac{(1 - \langle z_r, \omega_r \rangle)^{n+\alpha+1}}{(1 - |\omega_r|^2)^{(n+\alpha+1)/2}} = \frac{1}{K_{\omega_r}^{(\alpha)}(z_r)} \end{aligned} \tag{14}$$

Since  $\varphi_{\omega_r} \circ \varphi_{\omega_r} = id$ , we see that

$$\begin{aligned} \sum_{r=1}^n \left( U_{\omega_r}^{*(\alpha)} \left( U_{\omega_r}^{*(\alpha)} h_r \right) \right) (z_r) &= \sum_{r=1}^n \left( U_{\omega_r}^{*(\alpha)} h_r \right) (\varphi_{\omega_r}(z_r)) K_{\omega_r}^{(\alpha)}(z_r) \\ &= \sum_{r=1}^n h_r(z_r) K_{\omega_r}^{(\alpha)}(\varphi_{\omega_r}(z_r)) K_{\omega_r}^{(\alpha)}(z_r) = \sum_{r=1}^n h_r(z_r) \end{aligned} \tag{15}$$

for all  $z_r \in \mathbb{B}_n$  and  $h \in A_{\alpha}^2(\mathbb{B}_n)$ . Thus  $(U_{\omega_r}^{*(\alpha)})^{-1} = U_{\omega_r}^{*(\alpha)}$  and

$$T_{f \circ \varphi_{\omega_r}} U_{\omega_r}^{*(\alpha)} = U_{\omega_r}^{*(\alpha)} T_f \tag{16}$$

Holds for  $f \in L^{\infty} \mathbb{B}_n$

Proof : for  $h_r \in H^{\infty} \mathbb{B}_n$  and  $g \in A_{\alpha}^2(\mathbb{B}_n)$  we have

$$\begin{aligned} \sum_{r=1}^n \langle U_{\omega_r}^{*(\alpha)} T_f h_r, U_{\omega_r}^{*(\alpha)} g \rangle_{\alpha} &= \sum_{r=1}^n \langle T_f h_r, g \rangle_{\alpha} = \sum_{r=1}^n \langle f h_r, g \rangle_{\alpha} \\ &= \int_{B_n} \sum_{r=1}^n f(u) h_r(u) \overline{g(u)} dv_{\alpha}(z_r) \\ &= \int_{\mathbb{B}_n} \sum_{r=1}^n f(\varphi_{\omega_r}(z_r)) h_r(\varphi_{\omega_r}(z_r)) \overline{g(\varphi_{\omega_r}(z_r))} \left| K_{\omega_r}^{(\alpha)}(z_r) \right|^2 dv_{\alpha}(z) \\ &= \int_{B_n} \sum_{r=1}^n f(\varphi_{\omega_r}(z_r)) h_r(\varphi_{\omega_r}(z_r)) \overline{K_{\omega_r}^{(\alpha)}(z_r) g(\varphi_{\omega_r}(z_r)) K_{\omega_r}^{(\alpha)}(z_r)} dv_{\alpha}(z) \\ \sum_{r=1}^n \langle f U_{\omega_r}^{*(\alpha)} h_r, U_{\omega_r}^{*(\alpha)} g \rangle_{\alpha} &= \sum_{r=1}^n \langle T_{f \circ \varphi_{\omega_r}} U_{\omega_r}^{*(\alpha)} h_r, U_{\omega_r}^{*(\alpha)} g \rangle_{\alpha} \end{aligned}$$

Establishing (16).

## 2-necessary condition for boundedness

In this section we prove the following necessary condition for boundedness of the Toeplitz product  $T_f T_{\bar{g}}$  with  $f$  and  $g$  in  $A^2_\alpha(\mathbb{B}_n)$ .

**Theorem 1:** Let  $-1 < \alpha < \infty$ , and  $f$  and  $g$  in  $A^2_\alpha(\mathbb{B}_n)$ . If  $T_f T_{\bar{g}}$  is bounded on  $A^2_\alpha(\mathbb{B}_n)$  then

$$\sum_{r=1}^n \sup_{\omega_r \in \mathbb{B}_n} B_\alpha[|f|^2](\omega_r) B_\alpha[|g|^2](\omega_r) < \infty$$

Suppose  $f$  and  $g$  in  $A^2_\alpha(\mathbb{B}_n)$ .

Consider the operator

$f \otimes g$  on  $A^2_\alpha(\mathbb{B}_n)$  defined by

$$f \otimes g h_\alpha f,$$

For  $A^2_\alpha(\mathbb{B}_n)$ . It is easily proved that  $f \otimes g$  is bounded on  $A^2_\alpha(\mathbb{B}_n)$ . With norm equal to

$$\|f \otimes g\| = \|f\|_\alpha \|g\|_\alpha$$

We will obtain an expression for the operator  $f \otimes g$ , Where  $f$  and  $g$  in  $A^2_\alpha(\mathbb{B}_n)$ , in terms of Toeplitz product  $T_f T_{\bar{g}}$ , which we will be able to use to bound the norm of  $f \otimes g$  by a constant multiple of the norm of  $T_f T_{\bar{g}}$ , To obtain a suitable operator identity, we will use the Berezin transform :

Writing  $K_{\omega_r}^{(\alpha)}$  for the normalized reproducing Kernal, we define the Berezin transform of a bonded linear operator  $\mathcal{S}$  on  $A^2_\alpha(\mathbb{B}_n)$ . To be the function  $B_\alpha[\mathcal{S}]$  defined on  $\mathbb{B}_n$ .

$$\text{By } B_\alpha[\mathcal{S}](\omega_r) = \langle \mathcal{S} K_{\omega_r}^{(\alpha)}, K_{\omega_r}^{(\alpha)} \rangle_\alpha$$

For  $\omega_r \in \mathbb{B}_n$ , the boundedness of  $\mathcal{S}$  implies that the function  $B_\alpha[\mathcal{S}]$  is bounded on  $\mathbb{B}_n$ . The Berzin transform is injective, for  $B_\alpha[\mathcal{S}](\omega_r) = 0$  for all  $\omega_r \in \mathbb{B}_n$ , implies that  $\mathcal{S} = 0$ , the Zero operator on  $A^2_\alpha(\mathbb{B}_n)$  see [8] for a proof.

We will also make use of the following continuity condition of the Berezin transform: if  $\mathcal{S}_N \rightarrow \mathcal{S}$  in operator norm, then

$$B_\alpha[\mathcal{S}](\omega_r) = \lim_{N \rightarrow \infty} B_\alpha[\mathcal{S}_N](\omega_r),$$

for each  $\omega_r \in \mathbb{B}_n$ . The above statement is an immediate consequence of the following inequality:

$$|B_\alpha[\mathcal{S}](\omega_r) - B_\alpha[\mathcal{S}_N](\omega_r)| \leq \|\mathcal{S} - \mathcal{S}_N\|$$

To prove a suitable operator identity, we need the following lemma, which is inspired by [1, 2].

**Lemma 2:** Let  $-1 < \alpha < \infty$ . If  $\mathcal{S}$  is a bounded linear operator on  $A^2_\alpha(\mathbb{B}_n)$ . then

$$\left\| \sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z^\gamma} \mathcal{S} T_{\bar{z}^\gamma} \right\| \leq \|\mathcal{S}\|,$$

for every positive integer  $m$ .

**Proof:** Given a positive integer  $m$ , let  $d_m$  denote the number of elements of the set  $\{\gamma: |\gamma| = m\}$ .

Then  $\{\gamma: |\gamma| = m\} = \{\gamma_1, \gamma_2, \dots, \gamma_{d_m}\}$ . observe that each  $\gamma_k$  is a multi-index, that is,  $\gamma_k = (\gamma_{k,1}, \gamma_{k,2}, \dots, \gamma_{k,n}) \in \mathbb{N}^n$  with

$$\gamma_k = \gamma_{k,1} + \gamma_{k,2} + \dots + \gamma_{k,n} = m.$$

for each  $k$  in  $\{1, 2, \dots, d_m\}$  consider the monomials.

$$\sum_{r=1}^n \psi(z_r) = \sum_{r=1}^n \sqrt{\frac{m!}{\gamma_k!}} z_r^{\gamma_k} = \sum_{r=1}^n \sqrt{\frac{m!}{\gamma_k!}} z_{r_1}^{\gamma_{k,1}} z_{r_2}^{\gamma_{k,2}} \dots z_{r_n}^{\gamma_{k,n}}.$$

Define the row block operator

$$M_m = [T_{\psi_1}, T_{\psi_2}, \dots, T_{\psi_{d_m}}]: A_\alpha^2(\mathbb{B}_n) \times A_\alpha^2(\mathbb{B}_n) \times \dots \times A_\alpha^2(\mathbb{B}_n) \rightarrow A_\alpha^2(\mathbb{B}_n)$$

by

$$M_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} = \sum_{k=1}^{d_m} T_{\psi_k} f_k = P \left( \sum_{k=1}^{d_m} \psi_k f_k \right)$$

then

$$\begin{aligned} & \left\| M_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} \right\|^2 \leq \left\| \sum_{k=1}^{d_m} \psi_k f_k \right\|^2 \\ &= \int_{B_n} \sum_{r=1}^n \left| \sum_{k=1}^{d_m} \psi_k(z_r) f_k(z_r) \right|^2 dv_\alpha(z_r) \\ &\leq \int_{B_n} \sum_{r=1}^n \left( \sum_{k=1}^{d_m} |\psi_k(z_r)|^2 \right) \left( \sum_{k=1}^{d_m} |f_k(z_r)|^2 \right) dv_\alpha(z_r) \end{aligned}$$

Using the multinomial formula,

$$\sum_{r=1}^n \sum_{k=1}^{d_m} |\psi_k(z_r)|^2 = \sum_{k=1}^{d_m} \frac{m!}{\gamma_k!} |z^{\gamma_k}|^2 = \left( |z_{r_1}|^2 + |z_{r_2}|^2 + \dots + |z_{r_n}|^2 \right)^m |z_r|^{2m} \leq 1$$

Thus

$$\begin{aligned} & \left\| M_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} \right\| \leq \sum_{r=1}^n \left( \sum_{k=1}^{d_m} \int_{B_n} |f_k(z_r)|^2 dv_\alpha(z_r) \right)^{\frac{1}{2}} \\ &= \|(f_1, f_2, \dots, f_{d_m})\| \end{aligned}$$

If  $\mathcal{S}$  is a bounded linear operator on  $A_\alpha^2(\mathbb{B}_n)$ , the operator

$$\begin{aligned} \mathcal{S}: A_{\alpha}^2(\mathbb{B}_n) \times A_{\alpha}^2(\mathbb{B}_n) \times \dots \times A_{\alpha}^2(\mathbb{B}_n) \\ \rightarrow A_{\alpha}^2(\mathbb{B}_n) \times A_{\alpha}^2(\mathbb{B}_n) \times \dots \times A_{\alpha}^2(\mathbb{B}_n) \end{aligned}$$

defined by

$$\mathcal{S}_m \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d_m} \end{pmatrix} = \begin{pmatrix} S f_1 \\ S f_2 \\ \vdots \\ S f_{d_m} \end{pmatrix}$$

is bounded, in fact,

$$\|\mathcal{S}_m\| \leq \|\mathcal{S}\|$$

It is easily seen that the adjoint of  $M_m$  is the column block operator

$$M_m^* g = \begin{bmatrix} T_{\bar{\psi}_1} g \\ \vdots \\ T_{\bar{\psi}_{d_m}} g \end{bmatrix}$$

for  $A_{\alpha}^2(\mathbb{B}_n)$  we have

$$M_m \mathcal{S}_m M_m^* g = M_m \mathcal{S}_m \begin{bmatrix} T_{\bar{\psi}_1} g \\ \vdots \\ T_{\bar{\psi}_{d_m}} g \end{bmatrix} = M_m \begin{bmatrix} \mathcal{S} T_{\bar{\psi}_1} g \\ \mathcal{S} T_{\bar{\psi}_2} g \\ \vdots \\ \mathcal{S} T_{\bar{\psi}_{d_m}} g \end{bmatrix}$$

$$= \sum_{k=1}^{d_m} T_{\psi_k} (\mathcal{S} T_{\bar{\psi}_k} g) = \left( \sum_{k=1}^{d_m} T_{\psi_k} \mathcal{S} T_{\bar{\psi}_k} \right) g$$

So

$$M_m \mathcal{S}_m M_m^* = \sum_{k=1}^{d_m} T_{\psi_k} \mathcal{S} T_{\bar{\psi}_k}$$

That is

$$\sum_{r=1}^n \sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z_r \gamma} \mathcal{S} T_{\bar{z}_r \gamma} = M_m \mathcal{S}_m M_m^*.$$

It follows that

$$\sum_{r=1}^n \left\| \sum_{|\gamma|=m} \frac{m!}{\gamma!} T_{z_r \gamma} \mathcal{S} T_{\bar{z}_r \gamma} \right\| = \|M_m \mathcal{S}_m M_m^*\| \leq \|\mathcal{S}_m\| \leq \|\mathcal{S}\|$$

As was to be shown.

We will use the Berezin transform to derive an operator identity suitable for our purpose. It follows from (9) that

$$\sum_{r=1}^n \mathbb{B}_{\alpha}[\mathcal{S}](\omega_r) = \sum_{r=1}^n (1 - |\omega_r|^2)^{n+\alpha+1} \langle \mathcal{S} K_{\omega_r}^{(\alpha)}, K_{\omega_r}^{(\alpha)} \rangle_{\alpha},$$

for  $\omega_r \in \mathbb{B}_n$ . It is easily seen that

$$\sum_{r=1}^n T_{\bar{g}} K_{\omega_r}^{(\alpha)} = \sum_{r=1}^n \overline{g(\omega_r)} K_{\omega_r}^{(\alpha)}. \text{ Thus}$$

$$\begin{aligned} \sum_{r=1}^n \langle T_f T_{\bar{g}} K_{\omega_r}^{(\alpha)}, K_{\omega_r}^{(\alpha)} \rangle_{\alpha} &= \sum_{r=1}^n \langle T_{\bar{g}} K_{\omega_r}^{(\alpha)}, T_f K_{\omega_r}^{(\alpha)} \rangle_{\alpha} \\ &= \sum_{r=1}^n \langle \overline{g(\omega_r)} K_{\omega_r}^{(\alpha)}, f(\omega_r) K_{\omega_r}^{(\alpha)} \rangle_{\alpha} = \sum_{r=1}^n \frac{f(\omega_r) \overline{g(\omega_r)}}{(1 - |\omega_r|^2)^{n+\alpha+1}}, \end{aligned}$$

(73)

and we see that

$$\sum_{r=1}^n B_\alpha [T_f T_{\bar{g}}](\omega_r) = \sum_{r=1}^n f(\omega_r) \overline{g(\omega_r)}$$

we also have

$$\begin{aligned} & \sum_{r=1}^n B_\alpha [f \otimes g](\omega_r) (1 - |\omega_r|^2)^{n+\alpha+1} \langle (f \otimes g) K_{\omega_r}^{(\alpha)}, K_{\omega_r}^{(\alpha)} \rangle_\alpha \\ &= \sum_{r=1}^n (1 - |\omega_r|^2)^{n+\alpha+1} \langle \langle K_{\omega_r}^{(\alpha)}, g \rangle f, K_{\omega_r}^{(\alpha)} \rangle_\alpha \\ &= \sum_{r=1}^n (1 - |\omega_r|^2)^{n+\alpha+1} \langle K_{\omega_r}^{(\alpha)}, g \rangle \langle f, K_{\omega_r}^{(\alpha)} \rangle_\alpha \\ &= \sum_{r=1}^n (1 - |\omega_r|^2)^{n+\alpha+1} f(\omega_r) \overline{g(\omega_r)} \end{aligned}$$

We need the following lemma, which was proved in [11].

**Lemma 3:** If  $0 < \beta < 1$  and  $k$  is a positive integer, then

$$(1 - t)^{k-\beta} =$$

$$\sum_{j=0}^{k-1} (-1)^j \frac{\Gamma(k+1-\beta)}{\Gamma(k+1-\beta-j)! j!} t^j + (-1)^k \frac{\Gamma(k+1-\beta)}{\Gamma(\beta)\Gamma(1-\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+k)!} t^{n+k}$$

for all  $-1 < t < 1$ .

Assuming  $\alpha$  to be a non-integer, we apply the above lemma with  $k = [\alpha] + n + 2$  and  $\beta = 1 - [\alpha]$ , where  $[\alpha] = \alpha - \{\alpha\}$ , using that  $\Gamma([\alpha])\Gamma([1-\alpha]) = \frac{\pi}{\sin(\pi[\alpha])}$ , we get

$$\begin{aligned} (1 - t)^{n+1+\alpha} &= \sum_{j=0}^{n+1+[\alpha]} (-1)^j \frac{\Gamma(n+2-\alpha)}{j! \Gamma(n+2+\alpha-j)} t^j \\ &+ (-1)^{n+[\alpha]} \frac{\Gamma(n+2+\alpha) \sin(\pi[\alpha])}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(j+1-\{\alpha\})}{(n+2+[\alpha]+j)!} t^{n+2+[\alpha]+j} \end{aligned}$$

Note that the above formula is also valid in case  $\alpha$  is an integer. Applying identity to  $t = |\omega_r|^2 = \omega_1 \bar{\omega}_1, \dots, \omega_n \bar{\omega}_n$ ,

Making use of the multinomial formula see for example in [13].

We have

$$t^{n+2+[\alpha]+j} = \sum_{r=1}^n \sum_{|\gamma|=n+2+[\alpha]+j} \frac{t^{n+2+[\alpha]+j}}{\gamma!} \omega_r^\gamma \bar{\omega}_r^\gamma$$

and we obtain

$$\begin{aligned} & \sum_{r=1}^n (1 - |\omega_r|^2)^{n+1+\alpha} = \\ & \sum_{r=1}^n \sum_{j=0}^{n+1+[\alpha]} (-1)^j \sum_{|\gamma|=j} \frac{\Gamma(n+2+\alpha)}{\gamma! \Gamma(n+2+\alpha-j)} \omega_r^\gamma \bar{\omega}_r^\gamma \\ & + (-1)^{n+[\alpha]} \frac{(n+2+\alpha) \sin(\pi[\alpha])}{\pi} \times \sum_{n=0}^{\infty} \sum_{|\gamma|=t^{n+2+[\alpha]+j}} \frac{\Gamma(n+1-\alpha)}{\gamma!} \omega_r^\gamma \bar{\omega}_r^\gamma \end{aligned}$$

Combining the above formula with

$$\sum_{r=1}^n B_\alpha [f \otimes g](\omega_r) = \sum_{r=1}^n (1 - |\omega_r|^2)^{n+\alpha+1} f(\omega_r) \overline{g(\omega_r)}$$

$$\begin{aligned}
 & \sum_{r=1}^n B_{\alpha}[f \otimes g](\omega_r) \\
 &= \sum_{r=1}^n \sum_{j=0}^{n+1+[\alpha]} (-1)^j \sum_{|\gamma|=j} \frac{\Gamma(n+2+\alpha)}{\gamma! \Gamma(n+2+\alpha-j)} \omega^{\gamma} f(\omega_r) \overline{g(\omega_r)} \bar{\omega}^{\gamma} \\
 &+ (-1)^{n+[\alpha]} \frac{\Gamma(n+2+\alpha) \sin(\pi[\alpha])}{\pi} \\
 &\times \sum_{n=0}^{\infty} \sum_{|\gamma|=n+2+[\alpha]+j} \frac{\Gamma(j+1-[\alpha])}{\gamma!} \omega^{\gamma} f(\omega_r) \overline{g(\omega_r)} \bar{\omega}^{\gamma} \tag{17}
 \end{aligned}$$

Thus the above identity and uniqueness of the Berezin transform imply the following operator identity:

$$\begin{aligned}
 f \otimes g &= \sum_{r=1}^n \sum_{j=0}^{n+1+[\alpha]} (-1)^j \sum_{|\gamma|=j} \frac{\Gamma(j+1-\alpha)}{\gamma! \Gamma(n+2+\alpha-j)} T_{z_r^{\gamma}} T_f T_{\bar{g}} T_{\bar{z}_r^{\gamma}} \\
 &+ (-1)^{n+[\alpha]} \frac{\Gamma(n+2+\alpha) \sin(\pi[\alpha])}{\pi} \\
 &\times \sum_{n=0}^{\infty} \sum_{|\gamma|=n+2+[\alpha]+j} \frac{\Gamma(j+1-[\alpha])}{\gamma!} T_{z_r^{\gamma}} T_f T_{\bar{g}} T_{\bar{z}_r^{\gamma}} \tag{18}
 \end{aligned}$$

**Remark 4:** That the operator on the right-hand side of (18) defines a bounded operator follows from lemma 3 which implies

$$\begin{aligned}
 & \sum_{r=1}^n \sum_{n=0}^{\infty} \left\| \sum_{|\gamma|=n+2+[\alpha]+j} \frac{\Gamma(j+1-[\alpha])}{\gamma!} T_{z_r^{\gamma}} T_f T_{\bar{g}} T_{\bar{z}_r^{\gamma}} \right\| \leq \\
 & \sum_{j=0}^{\infty} \frac{\Gamma(j+1-[\alpha])}{(n+2+[\alpha]+j)!} \|T_f T_{\bar{g}}\|
 \end{aligned}$$

Stirling's formula shows that the series

$$\sum_{j=0}^{\infty} \frac{(\Gamma(j+1-[\alpha]))}{((n+2+[\alpha]+j)!)}$$

Converges. Thus

$$\sum_{r=1}^n \sum_{j=0}^{\infty} \sum_{|\gamma|=n+2+[\alpha]+j} \frac{\Gamma(j+1-[\alpha])}{\gamma!} T_{z_r^{\gamma}} T_f T_{\bar{g}} T_{\bar{z}_r^{\gamma}}$$

Is a bounded operator, which has Berezin transform equal to

$$\sum_{r=1}^n \sum_{j=0}^{\infty} \sum_{|\gamma|=n+2+[\alpha]+j} \frac{\Gamma(j+1-[\alpha])}{\gamma!} B_{\alpha} \left[ T_{z_r^{\gamma}} T_f T_{\bar{g}} T_{\bar{z}_r^{\gamma}} \right]$$

Thus the operator on the right-hand (18) has Berezin transform equal to the right-hand side of (17). By uniqueness of the Berezin transform the two operators must be equal, establishing operator identity (18).

We are now ready to prove the necessary condition of theorem1

**Proof of theorem 1:** Suppose  $f$  and  $g$  are analytic on  $A_{\alpha}^2(\mathbb{B}_n)$  such that the densely defined Toeplitz product  $T_f T_{\bar{g}}$  is bounded

on  $A_{\alpha}^2(\mathbb{B}_n)$ . Using identity (18) and the estimate in the above remark, we see that there exists a finite constant  $\bar{C}_{\alpha}$  such that

$$\|f \otimes g\| \leq \bar{C}_{\alpha} \|T_f T_{\bar{g}}\| \tag{19}$$

Thus

$$\|f\|_2 \|g\|_2 \leq \bar{C}_{\alpha} \|T_f T_{\bar{g}}\|$$

It follows from (16), applied to  $f$  and  $\bar{g}$ , that

$$\begin{aligned} \sum_{r=1}^n T_{f \circ \varphi_{\omega_r}} T_{\bar{g} \circ \varphi_{\omega_r}} &= \sum_{r=1}^n (T_{f \circ \varphi_{\omega_r}} U_{\omega_r}^{*(\alpha)}) U_{\omega_r}^{*(\alpha)} (T_{\bar{g} \circ \varphi_{\omega_r}} U_{\omega_r}^{*(\alpha)}) U_{\omega_r}^{*(\alpha)} \\ &= \sum_{r=1}^n (U_{\omega_r}^{*(\alpha)} T_f) U_{\omega_r}^{*(\alpha)} (U_{\omega_r}^{*(\alpha)} T_{\bar{g}}) U_{\omega_r}^{*(\alpha)} = \sum_{r=1}^n U_{\omega_r}^{*(\alpha)} (T_f T_{\bar{g}}) U_{\omega_r}^{*(\alpha)} \end{aligned}$$

for all  $\omega_r \in A_{\alpha}^2(\mathbb{B}_n)$ . In equality (19) applied to  $f \circ \varphi_{\omega_r}$  and  $g \circ \varphi_{\omega_r}$  gives

$$\sum_{r=1}^n \|f \circ \varphi_{\omega_r}\| \|g \circ \varphi_{\omega_r}\|_2 \leq \bar{C}_{\alpha} \sum_{r=1}^n \|T_{f \circ \varphi_{\omega_r}} T_{\bar{g} \circ \varphi_{\omega_r}}\| = \bar{C}_{\alpha} \|T_f T_{\bar{g}}\|$$

hence

$$\sum_{r=1}^n B_{\alpha}[|f|^2](\omega_r) B_{\alpha}[|g|^2](\omega_r) \leq \bar{C}_{\alpha}^2 \|T_f T_{\bar{g}}\|$$

for all  $\omega_r \in A_{\alpha}^2(\mathbb{B}_n)$ . So, for  $g \in A_{\alpha}^2(\mathbb{B}_n)$ , a necessary condition for the Toeplitz product  $T_f T_{\bar{g}}$  to be bounded on  $A_{\alpha}^2(\mathbb{B}_n)$  is

$$\sum_{r=1}^n \sup_{\omega_r \in \mathbb{B}_n} B_{\alpha}[|f|^2](\omega_r) B_{\alpha}[|g|^2](\omega_r) < \infty \tag{20}$$

This completes the proof of theorem 1.

### 3.Sufficient condition

In this section we will prove a condition slightly stronger than (20) in the following theorem.

**Theorem. 5:** Let  $-1 < \alpha < \infty$ , and let  $f$  and  $g$  be in  $A_{\alpha}^2(\mathbb{B}_n)$ . If for  $\varepsilon > 0$ .

$$\sum_{r=1}^n \sup_{\omega_r \in \mathbb{B}_n} B_{\alpha}[|f|^{2+\varepsilon}](\omega_r) B_{\alpha}[|g|^{2+\varepsilon}](\omega_r) < \infty, \tag{21}$$

Then the operator  $T_f T_{\bar{g}}$  is bounded on  $A_{\alpha}^2(\mathbb{B}_n)$ .

By Hölder's in equality,

$$\left( \int_{\mathbb{B}_n} |f|^2 dv_{\alpha} \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{B}_n} |f|^{2+\varepsilon} dv_{\alpha} \right)^{\frac{1}{(2+\varepsilon)}}$$

Applying this to the function  $f \circ \varphi_{\omega_r}$ , making use of (11), it follows that

$$\sum_{r=1}^n B_{\alpha}[|f|^2](\omega_r)^{\frac{1}{2}} \leq \sum_{r=1}^n B_{\alpha}[|f|^{2+\varepsilon}](\omega_r)^{\frac{1}{(2+\varepsilon)}}$$

and thus

$$(B_{\alpha}[|f|^2](\omega_r) B_{\alpha}[|g|^2](\omega_r))^{\frac{1}{2}} \leq (B_{\alpha}[|f|^{2+\varepsilon}](\omega_r) B_{\alpha}[|g|^{2+\varepsilon}](\omega_r))^{\frac{1}{(2+\varepsilon)}} \tag{22}$$

for all  $\omega_r \in \mathbb{B}_n$ , So condition (22) implies necessary condition (20).

In the proof that  $T_{\bar{f}}T_{\bar{g}}$  is bounded on  $A_{\alpha}^2(\mathbb{B}_n)$  if condition (22) holds, we need estimates on  $T_{\bar{f}}h_r$  and its derivatives, as alternative way to write the inner product formula in  $A_{\alpha}^2(\mathbb{B}_n)$ .

### 3.1. Two estimates.

We will need the two estimates conditioned in the following lemmas.

**Lemma 6:** Let  $-1 < \alpha < \infty$ . For  $f \in L^2(\mathbb{B}_n, v_{\alpha})$  and  $h_r \in H^{\infty}(\mathbb{B}_n)$  we have

$$\sum_{r=1}^n |(T_{\bar{f}}h_r)(\omega_r)| \leq \sum_{r=1}^n \frac{B_{\alpha}[|f|^2](\omega_r)^{\frac{1}{2}}}{(1-|\omega_r|^2)^{n+\alpha+1/2}} \|h_r\|_{\alpha}$$

for all  $\omega_r \in \mathbb{B}_n$ .

**Proof:** By Cauchy-Schwarz's inequality,

$$\begin{aligned} \sum_{r=1}^n |(T_{\bar{f}}h_r)(\omega_r)|^2 &\leq \sum_{r=1}^n \left( \int_{\mathbb{B}_n} \frac{|f(z_r)||h_r(z_r)|}{|1-\langle \omega_r, z_r \rangle|^{n+\alpha+1}} \right)^2 \\ &\leq \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{|f(z_r)|^2}{|1-\langle \omega_r, z_r \rangle|^{n+\alpha+1}} dv_{\alpha} \int_{\mathbb{B}_n} \sum_{r=1}^n |h_r(z_r)|^2 dv_{\alpha} \\ &= \sum_{r=1}^n \frac{B_{\alpha}[|f|^2](\omega_r)}{(1-|\omega_r|^2)^{n+\alpha+1}} \|h_r\|_{\alpha}^2 \end{aligned}$$

and the stated inequality follows.

**Lemma 7:** Let  $-1 < \alpha < \infty$  and  $\varepsilon > 0$ . For  $f \in L^2(\mathbb{B}_n, v_{\alpha})$ ,  $h_r \in H^{\infty}(\mathbb{B}_n)$ , and multi-index  $\gamma$  with  $|\gamma| = m \geq \frac{(n+\alpha+1)}{2}$  we have

$$\sum_{r=1}^n |(D^{\gamma}T_{\bar{f}}h_r)(\omega_r)| \leq \bar{C} \sum_{r=1}^n \frac{B_{\alpha}[|f|^{2+\varepsilon}](\omega_r)^{\frac{1}{(2+\varepsilon)}}}{(1-|\omega_r|^2)^m} \left( \int_{\mathbb{B}_n} \frac{|f(z_r)|^{\delta}}{|1-\langle \omega_r, z_r \rangle|^{n+\alpha+1}} dv_{\alpha}(z_r) \right)^{\frac{1}{\delta}}$$

for all  $\omega_r \in \mathbb{B}_n$ , where  $\delta = \frac{(2+\varepsilon)}{(1+\varepsilon)}$ .

**Proof :** Let  $\varepsilon > 0$ . For  $f \in L^2(\mathbb{B}_n, v_{\alpha})$  and  $h_r \in H^{\infty}(\mathbb{B}_n)$  we have

$$\begin{aligned} \sum_{r=1}^n (T_{\bar{f}}h_r)(\omega_r) &= \sum_{r=1}^n \langle T_{\bar{f}}h_r, K_{\omega_r}^{(\alpha)} \rangle_{\alpha} = \\ \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{\overline{f(z_r)}h_r(z_r)}{(1-\langle z_r, \omega_r \rangle)^{n+\alpha+1}} dv_{\alpha}(z_r), \omega_r &\in \mathbb{B}_n \end{aligned}$$

Thus

$$\sum_{r=1}^n (D^{\gamma}T_{\bar{f}}h_r)(\omega_r) = \frac{\Gamma(n+\alpha+m+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{\overline{z_r^{\gamma}f(z_r)}h_r(z_r)}{(1-\langle \omega_r, z_r \rangle)^{n+\alpha+m+1}} dv_{\alpha}(z_r)$$

for every multi-index  $\gamma$  with  $|\gamma| = m$ . Applying Hölder's inequality, we get

$$\begin{aligned} \sum_{r=1}^n |(D^\gamma T_{\bar{f}} h_r)(\omega_r)| &\leq C \left( \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{|f(z_r)|^{2+\varepsilon}}{|1-\langle \omega_r, z_r \rangle|^{2n+2\alpha+2}} \right)^{\frac{1}{2+\varepsilon}} \\ &= \bar{C} \sum_{r=1}^n \frac{B_\alpha[|f|^{2+\varepsilon}](\omega_r)^{\frac{1}{2+\varepsilon}}}{(1-|\omega_r|^2)^{\frac{n+\alpha+1}{1+\varepsilon}}} \\ &\times \left( \int_{\mathbb{B}_n} \frac{|h_r(z_r)|^\delta}{|1-\langle \omega_r, z_r \rangle|^{\frac{(2m+(n+\alpha+m+1)\varepsilon)}{1+\varepsilon}}} dv_\alpha(z_r) \right)^{\frac{1}{\delta}} \end{aligned}$$

Since  $2m \geq n + \alpha + 1$ , we have

$$\begin{aligned} &\sum_{r=1}^n |1 - \langle \omega_r, z_r \rangle|^{\frac{(2m+(n+\alpha+m+1)\varepsilon)}{1+\varepsilon}} \\ &\geq \sum_{r=1}^n (1 - |\omega_r|)^{\frac{(2m-n-\alpha-1+m\varepsilon)}{-\beta/(1+\varepsilon)}} |1 - \langle \omega_r, z_r \rangle|^{n+\alpha+1} \\ &\sum_{r=1}^n 2 \frac{(1 - |\omega_r|)^{\beta/(1+\varepsilon)}}{|1 - \langle \omega_r, z_r \rangle|^{n+\alpha+1}} \end{aligned}$$

Where  $\beta = 2m - n - \alpha - 1 + m\varepsilon$ , and thus

$$\begin{aligned} &\left( \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{|h_r(z_r)|^\delta}{|1-\langle \omega_r, z_r \rangle|^{\frac{(2m+(n+\alpha+m+1)\varepsilon)}{1+\varepsilon}}} dv_\alpha(z_r) \right)^{\frac{1}{\delta}} \\ &\leq \sum_{r=1}^n \frac{2^{\beta/(2+\varepsilon)}}{(1-|\omega_r|^2)^{\beta/2+\varepsilon}} \left( \int_{\mathbb{B}_n} \frac{|h_r(z_r)|^\delta}{|1-\langle \omega_r, z_r \rangle|^{n+\alpha+1}} dv_\alpha(z) \right)^{\frac{1}{\delta}} \end{aligned}$$

Hence

$$\begin{aligned} \sum_{r=1}^n |(D^\gamma T_{\bar{f}} h_r)(\omega_r)| &\leq \\ &\bar{C} \sum_{r=1}^n \frac{B_\alpha[|f|^{2+\varepsilon}](\omega_r)^{\frac{1}{2+\varepsilon}}}{(1-|\omega_r|^2)^{\frac{(n+\alpha+1)}{1+\varepsilon}}} \frac{2^{\beta/(2+\varepsilon)}}{(1-|\omega_r|^2)^{\beta/2+\varepsilon}} \left( \int_{\mathbb{B}_n} \frac{|f(z)|^\delta}{|1-\langle \omega, z_r \rangle|^{n+\alpha+1}} \right)^{\frac{1}{\delta}} \\ &\leq \bar{C} \sum_{r=1}^n \frac{B_\alpha[|f|^{2+\varepsilon}](\omega_r)^{\frac{1}{2+\varepsilon}}}{(1-|\omega_r|^2)^m} \left( \int_{\mathbb{B}_n} \frac{|h_r(z_r)|^\delta}{|1-\langle \omega_r, z_r \rangle|^{n+\alpha+1}} dv_\alpha(z) \right)^{\frac{1}{\delta}} \end{aligned}$$

This proves the stated inequality.

### 3.2.Inner product formula in $A_\alpha^2(\mathbb{B}_n)$ .

In this subsection we will establish a formula for the inner product in  $A_\alpha^2(\mathbb{B}_n)$  needed to prove our sufficiency condition for boundedness of Toeplitz products.

Let  $F$  and  $G$  be in  $A_\alpha^2(\mathbb{B}_n)$ . Then

$$(78)$$

$$\sum_{r=1}^n \Delta [F(z_r) \overline{G(z_r)}] = 4 \sum_{r=1}^n \sum_{j=1}^n D_j F(z_r) \overline{D_j G(z_r)},$$

and, by induction,

$$\sum_{r=1}^n \Delta^m [F(z_r) \overline{G(z_r)}] = 4^m \sum_{r=1}^n \sum_{|\gamma|=m} D^\gamma F(z_r) \overline{D^\gamma G(z_r)} \quad (23)$$

Where the sum is over all multi-indices  $\gamma$  with  $|\gamma| = m$ .

We have

$$\sum_{r=1}^n \Delta [(1 - |z_r|^2)^\beta] = 4\beta(\beta - 1) \sum_{r=1}^n (1 - |z_r|^2)^\beta - 4\beta(\beta + n - 1)(1 - |z_r|^2)^{\beta-1} \quad (24)$$

for all  $\beta \geq 2$ . To obtain an inner product formula suitable for the estimates we have established in the previous section, we will need the following lemma.

**Lemma8:** Let  $-1 < \alpha < \infty$  and let  $m$  be a positive integer. There exist constants  $a_1, a_2, \dots, a_{2m-1}$  and  $b_1, b_2, \dots, b_m$  (depending on  $n$  and  $\alpha$ ) such that

$$\begin{aligned} & \sum_{r=1}^n \Delta^m [(1 - |z_r|^2)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j (1 - |z_r|^2)^{\alpha+2m+j}] \\ &= \sum_{r=1}^n \left( 4^m \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)} (1 - |z_r|^2)^\alpha - \sum_{k=1}^{2m-1} b_k (1 - |z_r|^2)^{\alpha+2m+k-1} \right) \end{aligned} \quad (25)$$

**Proof:** Using (24) and induction it is easy to show that for every  $-1 < \alpha < \infty$  and positive integer  $m$ , there exist scalars  $\lambda_\alpha, o, \dots, \lambda_{\alpha,m}$  (depending on  $m$  and  $n$ ) such that

$$\sum_{r=1}^n \Delta^m [(1 - |z_r|^2)^{\alpha+2m}] = \sum_{r=1}^n \sum_{k=0}^m \lambda_{\alpha,k} (1 - |z_r|^2)^{\alpha+k} \quad (26)$$

Where  $\lambda_{\alpha,0} = 4^m \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)}$ . Fix a positive integer  $m$ ,

for convenience of notation, define  $\lambda_{\alpha,m} = o$  if  $j > m$  and  $\alpha$  is arbitrary.

From (26) we subtract  $\sum_{r=1}^n \sum_{j=1}^{2m-1} \Delta^m [a_j (1 - |z_r|^2)^{\alpha+2m+j}]$  for Scalars  $a_1, \dots, a_{2m-1}$ . That will be chosen such that all terms involving powers

$(1 - |z_r|^2)^{\alpha+k}$ , with  $1 \leq k \leq 2m - 1$ , drop out. To show that this can be done, we use (23) to note that

$$\begin{aligned} & \sum_{r=1}^n \sum_{j=1}^{2m-1} \Delta^m [a_j (1 - |z_r|^2)^{\alpha+2m+j}] = \\ & \sum_{r=1}^n \sum_{j=1}^{2m-1} \sum_{k=0}^m a_j \lambda_{\alpha,j,k} (1 - |z_r|^2)^{\alpha+k+j} \\ &= \sum_{r=1}^n \sum_{j=1}^{2m-1} \sum_{e=j}^{j+m} a_j \lambda_{\alpha+j,e-j} (1 - |z_r|^2)^{\alpha+l} \\ &= \sum_{r=1}^n \sum_{j=1}^{2m-1} \sum_{e=j}^{3m-1} a_j \lambda_{\alpha+j,e-j} (1 - |z_r|^2)^{\alpha+l} \end{aligned}$$

Using that  $\lambda_{\alpha+j,e-j} = 0$  if  $e > j + m$ . Interchanging the order of summation, we have

$$\begin{aligned} & \sum_{r=1}^n \sum_{j=1}^{2m-1} \Delta^m [a_j (1 - |z_r|^2)^{\alpha+2m+j}] = \\ & \sum_{r=1}^n \sum_{j=1}^{2m-1} \sum_{k=j}^{3m-1} a_j \lambda_{\alpha+j, k-j} (1 - |z_r|^2)^{\alpha+k} \\ & = \sum_{r=1}^n \sum_{k=1}^{2m-1} \sum_{j=1}^k a_j \lambda_{\alpha+j, k-j} (1 - |z_r|^2)^{\alpha+k} \\ & + \sum_{r=1}^n \sum_{k=2m}^{3m-1} \sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j, k-j} (1 - |z_r|^2)^{\alpha+k} \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{r=1}^n \Delta^m [a_j (1 - |z_r|^2)^{\alpha+2m}] - \sum_{r=1}^n \sum_{j=1}^{2m-1} a_j (1 - |z_r|^2)^{\alpha+2m+j} \\ & = \\ & \sum_{r=1}^n \lambda_{\alpha,0} (1 - |z_r|^2)^\alpha + \sum_{r=1}^n \sum_{k=1}^{2m-1} (\lambda_{\alpha,k} - \sum_{r=1}^n \sum_{j=1}^k a_j \lambda_{\alpha+j, k-j} (1 - |z_r|^2)^{\alpha+k} \\ & + \sum_{r=1}^n \sum_{k=2m}^{3m-1} \sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j, k-j} (1 - |z_r|^2)^{\alpha+k} \end{aligned}$$

Choose  $a_1, \dots, a_{2m-1}$  such that

$$\lambda_{\alpha,k} = \sum_{j=1}^k a_j \lambda_{\alpha+j, k-j}$$

for  $k = 1, \dots, 2m - 1$ . This can be done since  $(\lambda_{\alpha+j, i-1})_{i,j=1}^{2m-1}$  is a lower diagonal matrix with non-zero entries on its diagonal-putting

$$b_k = - \sum_{j=1}^{2m-1} a_j \lambda_{\alpha+j, 2m+k-1-j}$$

for  $k = 1, \dots, 2m$ , Eq (23) follows.

Let  $-1 < \alpha < \infty$  and assume that  $F, G \in A_\alpha^2(\mathbb{B}_n)$  are analytic on an open neighborhood of the closed unit ball  $\overline{\mathbb{B}}_n$ . Now, it is easily seen that

$$\Delta^k \sum_{r=1}^n [(1 - |z_r|^2)^{2\beta}] = \Delta^k \sum_{r=1}^n \left[ \frac{\delta}{\delta n} (1 - |z_r|^2)^{2\beta} \right] = 0$$

On  $\mathcal{S}$ , for  $k < \beta$  where  $\frac{\delta}{\delta n}$  denote the normal derivative. Repeatedly applying Green's formula and using lemma 6, we get

$$\begin{aligned} & \int_{\mathbb{B}_n} \sum_{r=1}^n \Delta^m [F\overline{G}](z_r) [(1 - |z_r|^2)^{2m} - \\ & \sum_{j=1}^{2m-1} a_j (1 - |z_r|^2)^{2m+j}] dv_\alpha(z_r) \\ & = \\ & \int_{\mathbb{B}_n} \sum_{r=1}^n \Delta^m [F\overline{G}](z_r) [(1 - |z|^2)^{\alpha+2m} - \\ & \sum_{j=1}^{2m-1} a_j (1 - |z_r|^2)^{\alpha+2m+j}] dv_\alpha(z_r) \\ & = \int_{\mathbb{B}_n} \sum_{r=1}^n F(z_r) \overline{G(z_r)} \Delta^m [(1 - |z_r|^2)^{\alpha+2m} - \sum_{j=1}^{2m-1} a_j (1 - \\ & |z_r|^2)^{\alpha+2m+j}] dv_\alpha(z_r) \\ & = \int_{\mathbb{B}_n} \sum_{r=1}^n F(z_r) \overline{G(z_r)} \end{aligned}$$

$$\begin{aligned} & \times \\ & \left[ 4^m \frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+1)} (1 - |z_r|^2)^\alpha - \sum_{j=1}^m b_j (1 - |z_r|^2)^{\alpha+2m+j-1} \right] dv_\alpha(z_r) \\ & = 4^m \frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+1)} \langle F, G \rangle_\alpha - \sum_{j=1}^m b_j \int_{\mathbb{B}_n} \sum_{r=1}^n F(z_r) \overline{G(z_r)} (1 - |z_r|^2)^{2m+j-1} dv_\alpha(z_r) \end{aligned}$$

It follows that

$$\begin{aligned} \langle F, G \rangle_\alpha & = 4^m \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \int_{\mathbb{B}_n} \sum_{r=1}^n \Delta^m [F\overline{G}](z_r) [(1 - |z_r|^2)^{2m} - \\ & \sum_{j=1}^{2m-1} a_j (1 - |z_r|^2)^{2m+j}] dv_\alpha(z_r) \\ & + \sum_{r=1}^n \sum_{j=1}^m b_j \int_{\mathbb{B}_n} F(z_r) \overline{G(z_r)} (1 - |z_r|^2)^{2m+j-1} dv_\alpha(z_r) \end{aligned}$$

Combining the above formula with (23), we obtain the following formula for the inner product in  $A_\alpha^2(\mathbb{B}_n)$  :

$$\begin{aligned} \langle F, G \rangle_\alpha & = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \\ & \sum_{|Y|=m} \int_{\mathbb{B}_n} \sum_{r=1}^n D^Y F(z_r) \overline{D^Y G(z_r)} (1 - |z_r|^2)^{2m} dv_\alpha(z_r) \\ & + \sum_{j=1}^{2m-1} a_j \sum_{|Y|=m} \int_{\mathbb{B}_n} \sum_{r=1}^n D^Y F(z_r) \overline{D^Y G(z_r)} (1 - |z_r|^2)^{2m} dv_\alpha(z_r) \\ & + \sum_{j=1}^m b_j \int_{\mathbb{B}_n} \sum_{r=1}^n F(z_r) \overline{G(z_r)} (1 - |z_r|^2)^{2m+j-1} dv_\alpha(z_r) \quad (27) \end{aligned}$$

for any  $m \in \overline{\mathbb{N}}$ . Given  $F, G \in A_\alpha^2(\mathbb{B}_n)$ , apply the above argument to the dilates  $F_r$  and  $G_r$  defined by  $F_r(z_r) = F(r \cdot z_r)$  and  $G_r = G(r \cdot z_r)$ , for  $|z_r| < Y_r$ , using that  $F_r \rightarrow F$  in  $A_\alpha^2(\mathbb{B}_n)$  and  $D^Y F_r \rightarrow D^Y F$  in  $A_\alpha^2(\mathbb{B}_n)$ , for every multi-index  $Y$  with  $|Y| = m$ , Eq (24) for general  $F, G \in A_\alpha^2(\mathbb{B}_n)$  is the limit of (27) for the dilates  $F_{r_0}$  and  $G_{r_0}$  as  $r \rightarrow 1^-$  in [13].

### 3.3. Sufficient condition for boundedness

The terms occurring in the inner product formula are all in  $L^2(\mathbb{B}_n, dM_\alpha)$ .

We are now in a position to prove our sufficiency condition for boundedness of Toeplitz products.

**Proof of theorem 5:** Assume that for  $\varepsilon > 0, M$  is a positive constant such that

$$B_\alpha[|f|^{2+\varepsilon}](\omega_r) B_\alpha[|g|^{2+\varepsilon}](\omega_r) \leq M^{2+\varepsilon}$$

for all  $\omega_r \in \mathbb{B}_n$ . By (26) we also have

$$B_\alpha[|f|^2](\omega_r) B_\alpha[|g|^2](\omega_r) \leq M^2$$

for all  $\omega_r \in \mathbb{B}_n$ . Let  $h$  and  $k$  be bounded analytic functions on  $\mathbb{B}_n$ . It follows from lemma 5 that

$$\sum_{r=1}^n |(T_{\bar{f}} h_r)(\omega_r)(T_{\bar{g}} k_r)(\omega_r)| \leq \sum_{r=1}^n \frac{M}{(1-|z_r|^2)^{n+\alpha+1}} \|h_r\|_{\alpha} \|k_r\|_{\alpha},$$

Thus

$$\int_{\mathbb{B}_n} \sum_{r=1}^n |(T_{\bar{f}} h_r)(\omega_r)(T_{\bar{g}} k_r)(\omega_r)| (1 - |z_r|^2)^q dv_{\alpha}(z_r) \leq M \|h_r\|_{\alpha} \|k_r\|_{\alpha}$$

for all  $q \geq n + \alpha + 1$ . So if we choose a large  $m$ , such that  $2m \geq n + \alpha + 1$ , then each of the terms

$$\int_{\mathbb{B}_n} \sum_{r=1}^n |(T_{\bar{f}} h_r)(\omega_r)(T_{\bar{g}} k_r)(\omega_r)| (1 - |z_r|^2)^{2m+j-1} dv_{\alpha}(z_r)$$

is bounded by  $\sum_{r=1}^n \bar{M} \|h_r\|_{\alpha} \|k_r\|_{\alpha}$ , for  $j=1, \dots, m$ .

Let  $Q_{\alpha}$  be the integral operator on  $L^2(\mathbb{B}_n, v_{\alpha})$ . defined by

$$Q_{\alpha} u \sum_{r=1}^n (\omega_r) = \int_{\mathbb{B}_n} \sum_{r=1}^n \frac{u(z_r)}{(1-|z_r|^2)^{n+\alpha-1}} dv_{\alpha}(z_r), \omega_r \in \mathbb{B}_n$$

Using lemma 6 for a multi-index  $\gamma$  with  $|\gamma| = m \geq \frac{(n+\alpha+1)}{2}$

We have

$$\left| (D^{\gamma} T_{\bar{g}} h)(\omega_r) \overline{(D^{\gamma} T_{\bar{f}} h_r)(\omega_r)} \right| \leq \bar{C}^2 \frac{M}{(1-|\omega_r|^2)^{2m}} (Q_{\alpha} |h_r|^{\delta}(\omega_r))^{\frac{1}{\delta}} (Q_{\alpha} |k|^{\delta}(\omega_r))^{\frac{1}{\delta}}$$

for all  $\omega_r \in \mathbb{B}_n$  where  $\delta = \frac{(2+\varepsilon)}{(1+\varepsilon)}$ . Since  $p = \frac{2}{\delta} > 1$  and  $Q_{\alpha}$  is  $L^p$ -bounded [13] there exists a constant  $\bar{N} > 0$ , such that

$$\int_{\mathbb{B}_n} \sum_{r=1}^n (Q_{\alpha} |h_r|^{\delta}(\omega_r))^{\frac{2}{\delta}} dv_{\alpha}(\omega_r) \leq \bar{N} \int_{\mathbb{B}_n} \sum_{r=1}^n (|h_r|^{\delta}(\omega_r))^{\frac{2}{\delta}} dv_{\alpha}(\omega_r) = \bar{N} \sum_{r=1}^n \|h_r\|_{\alpha}^2$$

and, likewise,  $\int_{\mathbb{B}_n} \sum_{r=1}^n (Q_{\alpha} |k_r|^{\delta}(\omega_r))^{\frac{2}{\delta}} dv_{\alpha}(\omega_r) \leq \bar{N} \|k_r\|_{\alpha}^2$

By the Cauchy-Schwarz inequality,

$$\int_{\mathbb{B}_n} \sum_{r=1}^n (Q_{\alpha} |h_r|^{\delta}(\omega_r))^{\frac{2}{\delta}} (Q_{\alpha} |k_r|^{\delta}(\omega_r))^{\frac{2}{\delta}} dv_{\alpha}(\omega_r) \leq \bar{N} \|h_r\|_{\alpha}^2 \|k_r\|_{\alpha}^2$$

We conclude that

$$\left| \int_{\mathbb{B}_n} \sum_{r=1}^n D^{\gamma} T_{\bar{g}} k_r(z_r) \overline{D^{\gamma} T_{\bar{f}} h(z_r)} (1 - |z_r|^2)^{2m+j} dv_{\alpha}(z_r) \right| \leq M \bar{N} \bar{C}^2 \|h\|_{\alpha} \|k_r\|_{\alpha}$$

for  $j = 0, 1, \dots, 2m - 1$ . Using inner product formula (25) with  $F = T_{\bar{g}} k_r$  and  $G = T_{\bar{f}} h$  we conclude that there is a finite constant  $L$  such that

$$\sum_{r=1}^n |\langle T_{\bar{f}} T_{\bar{g}} k_r, h_r \rangle_{\alpha}| \leq \sum_{r=1}^n L \|h_r\|_{\alpha} \|k_r\|_{\alpha}$$

for all bounded analytic functions  $h$  and  $k$  on  $\mathbb{B}_n$ . Hence the operator  $T_f T_{\bar{g}}$  is bounded on  $A_\alpha^2(\mathbb{B}_n)$ .

This completes the proof of **Lemma 8**

### 3.4. Compact Toeplitz products

The following theorem states that the Toeplitz product  $T_f T_{\bar{g}}$  is only compact in the trivial case that it is the zero operator.

**Theorem 9:** Let  $-1 < \alpha < \infty$ , and  $f$  and  $g$  be in  $A_\alpha^2(\mathbb{B}_n)$ .

Then  $T_f T_{\bar{g}}$  is compact if and only if  $f \equiv 0$  or  $g \equiv 0$ .

**Proof:** If  $T_f T_{\bar{g}}$  is compact on  $A_\alpha^2(\mathbb{B}_n)$ , then its Berezin transform Vanishes near the unit sphere  $\sum_{r=1}^n B_\alpha [T_f T_{\bar{g}}](\omega_r) \rightarrow 0$

as  $|\omega_r| \rightarrow 1^-$ . We have seen that  $\sum_{r=1}^n B_\alpha [T_f T_{\bar{g}}](\omega_r) = \sum_{r=1}^n f(\omega_r) \overline{g(\omega_r)}$ ,

$\sum_{r=1}^n |f(\omega_r) g(\omega_r)| = \sum_{r=1}^n |B_\alpha [T_f T_{\bar{g}}](\omega_r)| \rightarrow 0$

as  $|\omega_r| \rightarrow 1^-$ , and it follows from the Maximum Modulus Principle that  $fg \equiv 0$ .

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