SJBS (M), (2016b), 34: 23-29 ISSN: 7615-1858

Yang-Mills Equations Formulation of Euler –Lagrange equations.

Tagreed Ahmed[(1)] and Mohamed A. B[(2)]

Abstract:

Keywords: d operation, The Exterior derivative**,** Maxwell's equations .

INTRODUCTION:

Yang–Mills theory is a generalization of Maxwell's theory of electromagnetism, in which the basic dynamical variable is a connection on a G-bundle over 4-dimensional space-time. We discuss fibre bundle and G- bundles in particular .This G-bundle is the space on which our Yang-Mills theory will live, we present the definitions of connections and curvature in this setting, as well as the exterior covariant calculus .the mathematical formalism is applied to the case of Yang-Mills equation, in which is seen to reduce to Maxwell's equations in the case of an abelian gauge group.

1.1 The Euler-lagrange equations :-

Let G be a Lie group and P be a principal G-bundle over a d-dimensional manifold M.Recall that a gauge field is a connection on p .It is given by a 1- form A on M with values is the adjoint affine group $Ad(P) \subset End\ (g)$.

Here affine means that the transition functions belong to the affine group of the Lie algebra $\mathcal{g}$ of G, the transition function given by

(1) Sudan institute For natural Sciences .Khartoum-Sudan P.O.Box:3045.

(2) Mathematical Faculty of Sciences Technology EL-Neilain University. Khartoum-Sudan.

$$A_e = g.A_{\grave{e}}\, g^{-1} + g^{-1} d\, g$$

where A_e and $A_{\grave{e}}$ are the g valued $1-$ form on Mcorresponding to A Via two different trivialization of P (gauges) .The curvature of a gauge field A is a section F_A of the vector bundle $\wedge^2 \left(ad(E)\right) = \wedge^2 T^*(M) \otimes ad(E)$. It can be defined by the formula

$$F_A = dA + \frac{1}{2}[A, A]$$

If we fix a gauge, then $F = F_A$ is given by $n \times n$ matrix whose entries F_i^j are smooth 2-form on U, then we can write

$F_i^j = \sum_{\mu, \nu=1}^{d} F_{i,\mu\upsilon}^j(x)\, dx^\mu \wedge dx^\nu$.

Equivalently we can write

$$F = \left(F_{\mu\nu}\right) 1 \leq \mu , \nu \leq d,$$

Where $F_{\mu\nu}$ is the matrix with (ij)-entry equal to $F_{i,\mu\upsilon}^j(x)$ .We say that a gauge field A is a potential gauge field for F if $F = F_A$ .For example, let us take $G = U(1)$ and let P be trivial principal bundle $G-$ bundle. In this case $ad(P)$ is the trivial bundle $M \times R$ .So a gauge field is a 1- form

$\sum_{\mu=0}^{3} A_\mu\, dx^\mu$ which can be identified with a vector function

$$A = (A_\circ , A_1 , A_2 , A_3).$$

For any smooth function : $M \to U(1)$, the 1- form

$$\acute{A} = A + d \log \emptyset$$

defines the same connection. Its curvature is a smooth $2-$form

$$F = \sum_{\lambda , \nu=0}^{3} F_{\mu\upsilon}\, dx^\mu \wedge dx^\nu.$$

We shall identify it with the skew symmetric matrix $(F_{\mu\upsilon})$ whose entries are smooth functions in (t, x) :

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix} \qquad (1.1)$$

This is called the electromagnetic tensor. Since $[A,A] = 0$, we have

$F_{\mu\upsilon} = \partial_\mu A_\nu - \partial_\nu A_\mu.$ (1.2)

If we set

$$E = \left(E_1\ , E_{2\ ,} E_3\right), H = \left(H_1\ , H_{2\ ,} H_3\right)$$

Of course, equation(1.2) means that the differential form F_A satisfies

$$F_A = d\left(A_\circ\ dx^\circ + A_1\ dx^1 + A_2\ dx^2 + F_3\ dx^3\right)$$

Since R^4 is simply –connected, this occurs if and only if

$$\begin{aligned} dF = -E_1\, dx^\circ \wedge dx^1 - E_2\, dx^\circ \wedge dx^2 - E_3\, dx^\circ \wedge dx^3 + H_3\, dx^1 \wedge dx^2 \\ - H_2\, dx^1 \wedge dx^3 + H_1\, dx^2 \wedge dx^3 \\ = (-\partial_2 E_1 + \partial_1 E_2 + \partial_\circ H_3) dx^\circ \wedge dx^1 \wedge dx^2 \\ + (-\partial_3 E_1 + \partial_1 E_3 + \partial_\circ H_2)\, dx^\circ \wedge dx^1 \wedge dx^3 \\ + (-\partial_3 E_2 + \partial_1 E_3 + \partial_\circ H_3) dx^\circ \wedge dx^2 \wedge dx^3 \\ + div\, H dx^1 \wedge dx^2 \wedge dx^3 = 0. \end{aligned}$$

This is equivalent to

$$\nabla \times E + \frac{\partial H}{\partial t} = 0$$

$$\nabla . H = 0$$

This is first pair of Maxwell's equations. The second pair will follow from Euler-Lagrange equation for gauge fields.

2. The Yang-mills Lagrangian:

Is defined on the set of gauge fields. Its definition depends on the choice of a pseudo-Riemannian metric g on M which is locally given by

$$g = \sum_{\mu\ ,\nu=1}^{n} g_{\mu\upsilon}\ dx^\nu \wedge dx^\mu$$

Its value at a point $x \in M$ is a non-degenerate quadratic form on $T(M)_x$ is given by the inverse matrix $\left(g^{\mu\upsilon}(x)\right)$ .It defines a symmetric tensor

$$g^{-1} = \sum_{\mu\ ,\nu=1}^{n} g^{\mu\upsilon}\ \frac{\partial}{\partial x^\nu} \otimes \frac{\partial}{\partial x^\mu}$$

We can go to transform the g- valued 2- form $F = \left(F_{\mu\upsilon} \right)$ to the g- valued vector field

$$\hat{F} = (F^{\mu\upsilon}) = \left(g^{\mu\alpha} g^{\upsilon\beta} F_{\beta\alpha} \right) = \sum_{\mu,\nu=1}^{n} F^{\mu\upsilon} \frac{\partial}{\partial x^{\nu}} \otimes \frac{\partial}{\partial x^{\mu}}$$

Let $g \otimes g \rightarrow R$ be the Killing form on g .It is defined by

$$\langle A, B \rangle = -Tr\left(ad(A)^{\circ} ad(B)\right)$$

Where $ad: g \rightarrow End(g)$ is the ad-joint representation, $ad(A): X \rightarrow [A, X]$. This is a bilinear form which is invariant with respect to the ad joint representation $Ad: G \rightarrow GL(g)$. In the case when G is simi-simple, the Killing form is non-degenerate. Now we can form the scalar function

$$\langle F, \hat{F} \rangle := \sum_{\mu,\nu=1}^{n} \langle F_{\mu\upsilon}, F^{\mu\upsilon} \rangle \qquad (1.3)$$

This expression does not depend on the choice of coordinate functions. Now if we choose the volume form vol(g) on M associated to the metric g, we can integrate to get a functional on the set of gauge fields

$$S_{YM}(A) = \int_M \langle F, \hat{F} \rangle \, \text{vol}(g). \qquad (1.4)$$

This is called **the Yang-Mills action functional**.

2.1. Lemma: Let g be a metric on V and $\mu \in \wedge^n (V^*)$ be volume form associated to g,then there exists a unique linear isomorphism

$* : \wedge^k V^* \otimes W \rightarrow \wedge^{n-k} V^* \otimes W$, such that, for all $\alpha, \beta \in Hom(\wedge^k V, W)$,

$$\alpha \wedge * \beta = g^{-1}(\alpha, \beta)\mu \qquad (1.5)$$

Here $g^{-1}: V^* \times V^* \rightarrow R$ is the metric . Assume that we have a coordinate system where $g_{ij} = \delta \pm$ (a flat coordinate system). Then

$$g^{-1} = \left(d\,x^{i_1} \wedge \dots \wedge d\,x^{i_k}, d\,x^{j_1} \wedge \dots \wedge d\,x^{j_k} \right) = g^{i_1 j_1}, \dots . . g^{i_k j_k}.$$

This implies that

$$* \left(F_{i_1 \dots i_k} \, d\,x^{i_1} \wedge \dots \wedge d\,x^{i_k} \right) = g^{i_1 i_1}, \dots . . g^{i_k i_k} F_{i_1 \dots i_k} \, d\,x^{j_1} \wedge \dots \wedge d\,x^{j_{n-k}}, \qquad (1.6)$$

Where $d\,x^{i_1} \wedge \dots \wedge d\,x^{i_k} \wedge d\,x^{j_1} \wedge \dots \wedge d\,x^{j_{n-k}} = dx^1 \wedge \dots \wedge dx^n$

If $g^{ij} = \delta_{ij}$ then this formula can be written as

$$(*F)_{j_1\ldots\ldots j_k} = \in \left(i_{1,\ldots\ldots}, i_{k\,,}\ j_{1,\ldots\ldots}, j_{n-k}\right) F_{i_1\ldots\ldots i_k}\, d\,x^{j_1} \wedge \ldots \wedge d\,x^{j_{n-k}}.$$

Where $\in \left(i_{1,\ldots\ldots}, i_{k\,,}\ j_{1,\ldots\ldots}, j_{n-k}\right)$

is the sign of the permutation $\left(i_{1,\ldots\ldots}, i_{k\,,}\ j_{1,\ldots\ldots}, j_{n-k}\right)$.If G is compact semi-simple group we may consider the unitary inner product in the space $\mathcal{A}^k\big(ad(P)\big)(M)$

$$\langle F_1\,,\ F_2\rangle = \int_M F_1 \wedge * F_2. \qquad (1.7)$$

2.2. Lemma: Assume M is compact, or F, G vanish on boundary of M.Let A be a connection on P .Then, for any $F_1 \in \mathcal{A}^k\big(ad(P)\big)(M)$, $F_2 \in \mathcal{A}^{k+1}\big(ad(P)\big)(M)$

$$d^A\langle F_1\,,\ F_2\rangle = (-1)^{k+1}\langle F_1\,, d^A *\ F_2\rangle.$$

3. The Euler-Lagrange equations for the Yang-Mills action functional:

Where the norm is taken in the sense of (1.7) .We know that the difference of two connections is a 1- form

$$S_{YM}\,(A) = \int_M \langle F_A, \tilde{F}_A\rangle\, \mathrm{vol}(g) = \int_M F_1 \wedge * F_2 = \|F_A\|^2,$$

Where the norm is taken in the sense of (1.7) .We know that the difference of two connections is a 1- form with values in $ad(P)$

, for any $h \in \mathcal{A}^1\big(ad(P)\big)$ and $\in \Gamma(E)$,

$$F_{A+h} = d(A+h) + \frac{1}{2}[A+h\,, A+h] = F_A + dh + [A, h] + 0(\|h\|)$$

$$= F_A + d^A(h).$$

Now, ignoring terms of order $0(\|h\|)$,we get

$$S_{YM}\,(A+h) - S_{YM}\,(A) = \|F_{A+h}\|^2 - \|F_A\|^2 = \|F_{A+h} - F_A\|^2 +$$

$$2\langle F_A\,, F_{A+h} - F_A\rangle = 2\int_M d^A h \wedge * F_A = -2\int_M h \wedge d^A(*F_A).$$

The last step, we used lemma2.This implies the equation for a critical connection :

$$d^A(*F_A) = 0 \qquad (1.8)$$

It is called **the Yang-Mills equation**, by Bianchi s identity, we always have $d^A(*F_A) = 0$

3.1 Definition: A connection satisfying the Yang-Mills equation (1.8) is called **the Yang-Mills equation connection**.

3.2 Definition. An instanton is a principal $SU(2)$ -bundle P over a four dimensional Riemannian manifold with $k\big(ad(P)\big) > 0$ together with an anti-self- dual connection. The number $k\big(ad(P)\big)$ is called the instanton number.

Note that . all complex vector G- bundles of the same rank and the same Chern classes are isomorphic .So, Let us fix one principal $SU(2)$ -bundle P with given instanton number k and consider the set of all anti-self –dual connections on it .The group of gauge transformations acts naturally on this set, we can consider the moduli space

$\mu_k\,(M) = \{\text{anti} - \text{self} - \text{dual connections on } P\}/\mathcal{G}$.

, when M is a smooth structure on a nonsingular algebraic surface, the moduli space of stable holomorphic rank 2 bundles with first Chern class zero and second Chern class equal to k.

Let us take $M = \mathbb{R}^4$. Take the Lorentzian metric g on M. Let $F = dA = \sum\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)\, dx^\mu \wedge dx^\nu$,

Then

$$* F = -F_{23}\, dx^\circ \wedge dx^1 - F_{01}\, dx^2 \wedge dx^3 + F_{31}\, dx^\circ \wedge dx^2 - F_{02}\, dx^3 \wedge dx^1 + F_{12}\, dx^0 \wedge dx^3 - F_{103}\, dx^1 \wedge dx^2 =$$
$$H_1\, dx^\circ \wedge dx^1 + E_1\, dx^2 \wedge dx^3 + H_2\, dx^\circ \wedge dx^2 + E_2\, dx^3 \wedge dx^1 + H_3\, dx^0 \wedge dx^3 + E_3\, dx^1 \wedge dx^2.$$

It corresponds to the matrix

$$* F = \begin{pmatrix} 0 & -H_1 & H_2 & H_3 \\ -H_1 & 0 & E_3 & -E_2 \\ -H_2 & -E_3 & 0 & E_1 \\ -H_3 & E_2 & -E_1 & 0 \end{pmatrix} \qquad (1.9)$$

(H, iE). Equivalently, if we form the complex electromagnetic tensor $+iE$, then $* F$ corresponds to $i(H + iE)$ .The Yang-Mills equation

$$0 = d^A(* F) = d(* F)$$

Gives the second pair of Maxwell equations(in vacuum)

$$\nabla \times H - \frac{\partial E}{\partial t} = 0$$

$\nabla . E = 0$.

3. 3 Remark: The two equation $dF = 0$ and $d(*F) = 0$ can be stated as one equation $(dd^* + d^*d)F = 0$ .Where d^* is operator adjoint to the operator d with respect to the unitary metric on the space of forms defined by (1.7) .

References:

[1] A. Cattaneo, G. Felder, A path integral approach to the kontsevich quantization formula, Comm. Math. Phys.212 (2000).

[2] A.S. Cattaneo, G. Fedler, Poisson sigma models and symplectic groupoids, in: N.P. Landsman, M. Pflaum, M. Schlichenmaier (Eds.), Quantization of Singular Symplectic Quotients, Birkhäuser, Basel, Boston, Berlin, 2001.

[3] Alexei Kotov, Thomas Strobl, Characteristic Classes Associated to q-Bundles, 2007.

[4] A. Cannas da Silva, A. Weinstein, Geometric Models for Non-commutative Algebras, in: Berkeley Mathematics Lecture Notes, AMS, 1999.

[5] C. Boyer, B. Mann, J. Hurtubise and R. Milgram The topology of instanton moduli spaces, I: the Atiyah-Jones conjecture Annals of Math. 137 (1993).

[6] C.A. Rossi, The division map of principal bundles with groupoid structure and generalized gauge transformations. 2004 .

[7] Conlon, L. Differentiable Manifolds, 2nd ed., Birkh¨auser, Boston, 2001.

[8] E. Calabi Extremal Kahler metrics In:Seminar in Differential geometry (ed. Yau) Princeton U.P. (1983).

[9] F. Bonechi, M. Zabzine, Poisson sigma model over group manifolds, J. Geom. Phys. 54 (2005) 173_196.