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Asymmetric multi-channel sampling in a series of shift invariant spaces

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Abstract:

We show asymmetric multi-channel sampling on a series of a shift invariant spaces $\sum_{d=1}^{m} V\ (\varphi(t_d))$ with a series of Riesz generators $\sum_{d=1}^{m} \varphi(t_d)$ in $L^2(\mathbb{R})$, where each channeled signal is assigned a uniform but distinct sampling rate. We use Fourier duality between $\sum_{d=1}^{m} V\ (\varphi(t_d))$ and $L^2[0,2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on $\sum_{d=1}^{m} V\ (\varphi(t_d))$.

Keywords: Shift invariant space, Multi-channel sampling , Frame Riesz basis

1. Introduction:

The multi-channel sampling method goes back to the works of Shannon [18] and Fogel [7], where reconstruction of a band-limited signal from samples of the signal and its derivatives was found. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [16]. Since Papoulis' fundamental work, there have been many generalizations and applications of multi-channel sampling. See [1,5,6,14,17,19].

Papoulis' result has also been extended to a general shift invariant space by using the filter banks technique (see [4,19,20]). More recently Garcia and Pérez-Villalon [8] derived

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stable generalized sampling in a shift invariant space. Most previous work related to multi-channel sampling has assumed that the sampling rates of all channels are the same.

S. Kang, J.M. Kim, K.H. Kwon [22] considered asymmetric multi-channel sampling in a shift invariant space $V(\varphi)$with a suitable Riesz generator $\varphi(t)$, where each channeled signal is sampled with a uniform but distinct rate.Using Fourier duality between $\sum_{d=1}^{m} V(\varphi(t_d))$ and $L^2[0,2\pi]$ [8,9,10,22], we derive under the same considerations a stable series of shifted asymmetric multi-channel sampling formula in $\sum_{d=1}^{m} V(\varphi(t_d))$. The corresponding symmetric multi-channel sampling in $\sum_{d=1}^{m} V(\varphi(t_d))$ was handled in [9],[22], where $\sum_{d=1}^{m} \varphi(t_d)$ is a continuous series of Riesz generators satisfying

$\sup_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi(t_d - n)|^2 < \infty$. In this case all signals in $\sum_{d=1}^{m} V(\varphi(t_d))$ are continuous on $\mathbb{R}$ [21],[22]. We require only that the series of Riesz generators $\sum_{d=1}^{m} \varphi(t_d)$ are pointwise well defined everywhere on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi(t_d - n)|^2 < \infty, \sum_{d=1}^{m} t_d \in \mathbb{R}$. Hence we essentially allow any series of Riesz generators in $L^2(\mathbb{R})$. Hence [22] allow more general filters than the ones in [8] by asking only that the impulse responses of filters belong to $L^2(\mathbb{R})$ (or the frequency responses of filters belong to

$L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$when $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0,2\pi]$), whereas they belong to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in [9]. We give an illustrative example (see[22]).

2. Preliminaries:

We consider the notations and formulas in [22]. The normalized Fourier transform is

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^{m} \varphi(t_d) \prod_{d=1}^{m} e^{-it_d\xi}\, dt_d , \sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\frac{1}{\sqrt{2\pi}} \mathcal{F}[\cdot]$ extends to a unitary operator from $L^2(\mathbb{R})$onto $L^2(\mathbb{R})$. For any $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$, let

$$\sum_{d=1}^{m} C_{\varphi}(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi(t_d + n)|^2 \text{ and } G_{\varphi}(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2.$$

Then

$$\sum_{d=1}^{m} C_{\varphi}(t_d) = \sum_{d=1}^{m} C_{\varphi}(t_d + 1) \in L^1[0,1], G_{\varphi}(\xi) = G_{\varphi}(\xi + 2\pi) \in L^2[0,2\pi]$$

and

$$\left\|\sum_{d=1}^{m} \varphi(t_d)\right\|^2_{L^{2}(\mathbb{R})} = \left\|\sum_{d=1}^{m} C_{\varphi}(t_d)\right\|_{L^1[0,1]} = \frac{1}{2\pi} \|G_{\varphi}(\xi)\|_{L^1[0,2\pi]}.$$

In particular, $\sum_{d=1}^{m} C_{\varphi}(t_d) < \infty$ for a.e. $\sum_{d=1}^{m} t_d \in \mathbb{R}$. We also let

$$\sum_{d=1}^{m} Z_{\varphi}(t_d, \xi) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} \varphi(t_d + n) e^{-in\xi}$$

be the Zak transform [12] of $\sum_{d=1}^{m} \varphi(t_d)$ in $L^2(\mathbb{R})$). Then $\sum_{d=1}^{m} Z_{\varphi}(t_d, \xi)$ is well defined a.e. on $\mathbb{R}^2$ and is quasi-periodic in the sense that

$\sum_{d=1}^{m} Z_{\varphi}(t_d + 1, \xi) = e^{i\xi} \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi)$and $\sum_{d=1}^{m} Z_{\varphi}(t_d, \xi + 2\pi) = \sum_{d=1}^{m} Z_{\varphi}(t_d, \xi)$.

A Hilbert space H consisting of complex valued functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if there is a series of a functions $\sum_{d=1}^{m} q(s, t_d)$ on $E \times E$, called the reproducing kernel of H, satisfying

(i) $\sum_{d=1}^{m} q(., t_d) \in H$ for each $\sum_{d=1}^{m} t_d \in E$,

(ii) $\langle f(s), \sum_{d=1}^{m} q(s, t_d)\rangle = \sum_{d=1}^{m} f(t_d), f \in H$.

In an RKHS H, any norm converging sequence also converges uniformly on any subset of E, on which $\| \sum_{d=1}^{m} q(., t_d) \|^2_H = \sum_{d=1}^{m} q(t_d, t_d)$ is bounded.

A sequence $\{\varphi_n : n \in \mathbb{Z}\}$ of vectors in a separable Hilbert space H is

(i) a Bessel sequence with a bound $A + \epsilon : \epsilon > 0$ if

$$\sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \leq (A + \epsilon)\|\varphi\|^2, \varphi \in H, \epsilon > 0,$$

(ii) a frame of H with bounds $A+\epsilon \geq A : \epsilon > 0$ if

$$A\|\varphi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \le (A+\epsilon)\|\varphi\|^2, \varphi \in H, \epsilon > 0,$$

(iii) a Riesz basis of H with bounds $A+\epsilon \geq A : \epsilon > 0$ if it is complete in H and

$$A\|\boldsymbol{c}\|^{\mathbf{2}} \le \left\|\sum_{n\in\mathbb{Z}} c(n)\varphi_n\right\|^2 \le (A+\epsilon)\|c\|^2, c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2, \epsilon > 0,$$

where $\|\boldsymbol{c}\|^{\mathbf{2}} = \sum_{n\in\mathbb{Z}} |\mathrm{c(n)}|^2$.

We let $\sum_{d=1}^{m} V\,(\varphi(t_d))$ be the series of the shift invariant spaces, where $\sum_{d=1}^{m} \varphi(t_d)$ is a series of a Riesz generators, that is, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^{m} V\,(\varphi(t_d))$. Then

$$\sum_{d=1}^{m} V\left(\varphi(t_d)\right) = \left\{\sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} c(n)\varphi(t_d - n) : C = \{c(n)\}_{n\in\mathbb{Z}} \in l^2\right\}.$$

It is well known see [2] that $\sum_{d=1}^{m} \varphi(t_d)$ is a series of Riesz generators if and only if there is a constant A such that $A \le G_\varphi(\xi) \le A + \epsilon \;\; a.e. on\, [0, 2\pi]$. In this case, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^{m} V\,(\varphi(t_d))$ with bound $\epsilon > 0$. We assume further that

(i) $\sum_{d=1}^{m} \varphi(t_d)$ is everywhere well defined on $\mathbb{R}$;

(ii) $\sum_{d=1}^{m} C_\varphi(t_d) < \infty, \sum_{d=1}^{m} t_d \in \mathbb{R}, i.e., \{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\} \in l^2$ for each $\sum_{d=1}^{m} t_d \in \mathbb{R}$.

We then allow essentially all series of Riesz generators since for any $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R}), \sum_{d=1}^{m} C_\varphi(t_d) < \infty$ a.e. so that $\sum_{d=1}^{m} \varphi(t_d)$ has an equivalent representative satisfying the above two conditions. Then for each $\boldsymbol{c} \in l^2, \sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d)$ converges both in $L^2(\mathbb{R})$and absolutely for each $\sum_{d=1}^{m} t_d \in \mathbb{R}$. Hence $\sum_{d=1}^{m} V(\varphi(t_d))$ becomes an RKHS with the reproducing kernel (see [13])

$\sum_{d=1}^{m} q(s,t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\tilde{\varphi}(s-n)\overline{\varphi(t_d-n)}$, where $\left\{\sum_{d=1}^{m}\tilde{\varphi}(t_d - n) : n \in \mathbb{Z}\right\}$

is the series of the dual Riesz bases of $\{\sum_{d=1}^{m}\varphi(t_d-n) : n \in \mathbb{Z}\}$ with bounds for $\epsilon > 0$. As in [9,10], we introduce an isomorphism $\mathcal{J}$ from $L^2[0,2\pi]$ onto $\sum_{d=1}^{m} V(\varphi(t_d))$ defined as:

$$\begin{aligned}\sum_{d=1}^{m}(\mathcal{J}\,F\,)(t_d) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\langle F\,(\xi\,), e^{-in\xi}\rangle_{L^2[0,2\pi]}\,\varphi(t_d\,-n)\\ &= \sum_{d=1}^{m}\langle F\,(\xi\,), \frac{1}{2\pi}Z_{\varphi}(t_d,\xi)\rangle_{L^2[0,2\pi]}.\end{aligned}$$

We then have:

(i) $(\widehat{\mathcal{J}F}\,)(\xi\,) = F\,(\xi\,)\,\hat{\varphi}(\xi)$

(ii) $\mathcal{J}\,(F\,(\xi\,)e^{-in\xi}\,) = \sum_{d=1}^{m}(\mathcal{J}\,F\,)(t_d\,-n)\,, n \in \mathbb{Z}$.

3. Asymmetric multi-channel sampling

The aim of this paper is as follows (see [22]). Let $\{L_{(1+\epsilon_1)}[\cdot] : \epsilon_1 \geq 0\}$ be N LTI (linear time-invariant) systems with impulse responses $\{\sum_{d=1}^{m} L_{(1+\epsilon_1)}(t_d) : \epsilon_1 \geq 0\}$. Develop a stable series of shifted multi-channel sampling formula for any signal $\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V\,(\varphi(t_d))$ using discrete sample values from $\{\sum_{d=1}^{m} L_{(1+\epsilon_1)}(t_d) : \epsilon_1 \geq 0\}$, where each channeled signal $\sum_{d=1}^{m} L_{(1+\epsilon_1)}\,[f\,](t_d)$ for $\epsilon_1 \geq 0$ is assigned with a distinct sampling rate

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} L_{(1+\epsilon_1)}\,[f\,]\big(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}n\big)s_{(1+\epsilon_1),n}\,(t_d)\,,$$

$$\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V\,(\varphi(t_d))\,, \qquad (1)$$

where $\left\{\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : \epsilon_1 \geq 0, n \in \mathbb{Z}\right\}$ is a series of frames or a Riesz basis of $\sum_{d=1}^{m} V\,\big(\varphi(t_d)\big)$,

$\{(1+\epsilon_2)_{(1+\epsilon_1)} : \epsilon_1 \geq 0\}$ are positive integers, and $\{\sigma_{(1+\epsilon_1)} : \epsilon_1 \geq 0\}$ are real constants. Note that the series of shifting of sampling instants is unavoidable in some uniform sampling [12] and arises naturally when we allow rational sampling periods in (1). Here, we assume that each $L_{(1+\epsilon_1)}\,[\cdot]$ is one of the following three types: the impulse response $\sum_{d=1}^{m} l(t_d)$ of an LTI system is such that

(i) $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d + a), a \in \mathbb{R}$ or

(ii) $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R})$ or

(iii) $\hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R})$ when $H_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)| \in L^2[0,2\pi]$. For type (i),

$\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} f(t_d + a), f \in L^2(\mathbb{R})$ so that $L[\cdot]: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isomorphism. In particular, for any $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m} V\left(\varphi(t_d)\right)$,

$\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \psi)(t_d)$ converges absolutely on $\mathbb{R}$ since

$$\sum_{d=1}^{m} C_\psi(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\psi(t_d + n)|^2 < \infty, \sum_{d=1}^{m} t_d \in \mathbb{R}, \text{where}$$

$\sum_{d=1}^{m} \psi(t_d) = \sum_{d}^{m} L[\varphi](t_d) = \sum_{d=1}^{m} \varphi(t_d + a)$. For types (ii) and (iii), we have the following results (see [22]):

Lemma 3.1. Let $L[\cdot]$ be an LTI system with the impulse response $\sum_{d=1}^{m} l(t_d)$ of the type (ii) or (iii) as above and

$\sum_{d=1}^{m} \psi(t_d) = \sum_{d=1}^{m} L[\varphi](t_d) = \sum_{d=1}^{1} (\varphi * l)(t_d)$. Then

(a) $$\sum_{d=1}^{m} \psi(t_d) \in C_\infty(\mathbb{R})$$

$$= \left\{ \sum_{d=1}^{m} u(t_d) \in C(\mathbb{R}) : \lim_{\sum_{d=1}^{m} |t_d| \to \infty} \sum_{d=1}^{m} u(t_d) = 0 \right\},$$

(b) $sup_{\mathbb{R}} \sum_{d=1}^{m} C_\psi(t_d) < \infty$;

(c) for any $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m} V\left(\varphi(t_d)\right)$,

$\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \psi)(t_d)$ converges absolutely and uniformly on $\mathbb{R}$.

Hence $\sum_{d=1}^{m} L[f](t_d) \in C(\mathbb{R})$.

Proof: First assume $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R})$. Then $\sum_{d=1}^{m} \psi(t_d) \in C_\infty(\mathbb{R})$ by the Riemann–Lebesgue lemma since $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^1(\mathbb{R})$. Since

$$\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right|$$

$$\le G_\varphi(\xi)^{\frac{1}{2}} G_l(\xi)^{\frac{1}{2}} \quad ,$$

$$\left\|\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(\xi\ +2n\pi)\right|\right\|^2_{L^2[0,2\pi]} \leq \int_0^{2\pi} G_\varphi(\xi\ )\,G_l(\xi\ )d\xi$$

$$\leq 2\pi\left\|G_\varphi(\xi\ )\right\|_{L^\infty(\mathbb{R})}\|l\|^2{}_{L^2(\mathbb{R})}.$$

Thus for any $\sum_{d=1}^{m} t_d$ in $\mathbb{R}$, we have by the Poisson summation formula (se [13])

$$\sum_{n\in\mathbb{Z}}\hat{\psi}(\xi\ +2n\pi)\prod_{d=1}^{m} e^{it_d(\xi+2n\pi)}$$

$$=\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\psi(t_d\ +n)\,e^{-in\xi}\ \text{in}\ \ L^2\ [0,2\pi]$$

Therefore for any $\sum_{d=1}^{m} t_d$ in $\mathbb{R}$

$$\sum_{d=1}^{m} C_\psi(t_d)\ =\ \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}|\psi(t_d\ +n)|^2$$

$$=\frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\psi(t_d\ +n)\,e^{-in\xi}\right\|^2_{L^2\ [0,2\pi]}$$

$$=\ \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}}\hat{\psi}(\xi\ +2n\pi)\prod_{d=1}^{m} e^{it_d(\xi+2n\pi)}\right\|^2_{L^2\ [0,2\pi]}$$

$$\leq\ \left\|G_\varphi(\xi\ )\right\|_{L^\infty(\mathbb{R})}\|l\|^2{}_{L^2(\mathbb{R})}.$$

By Young's inequality on the convolution product, $\|L[\,f\,]\|_{L^\infty(\mathbb{R})}\ \leq \|f\ \|_{L^2(\mathbb{R})}\|l\|\ _{L^2(\mathbb{R})}$ so that $L[\cdot]:\ L^2(\mathbb{R})\to L^\infty(\mathbb{R})$ is a bounded linear operator. Hence for any

$$\sum_{d=1}^{m} f(t_d)\ =\ \sum_{d=1}^{m}(c\ *\ \varphi)(t_d)$$

$$=\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} c(n)\varphi(t_d\ -n)\ \in\ \sum_{d=1}^{m} V\ (\varphi(t_d)),$$

$$\sum_{d=1}^{m} L[\,f\,](t_d)\ =\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} c(n)L[\varphi(t_d\ -n)]\ =\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} c(n)\psi(t_d\ -n),$$

which converges absolutely and uniformly on R by (b). Now assume that $H_\varphi(\xi\ )\ \in\ L^2\ [0,2\pi]$. The case $\hat{l}(\xi\ )\ \in\ L^2(\mathbb{R})$ is reduced to type (ii). So let $\hat{l}(\xi) \in L^{\infty}(\mathbb{R})$. Then $\hat{\varphi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and so $\psi(\xi) \in C_{\infty}(R) \cap L^2(\mathbb{R}))$. Since

$$\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right|$$

$\leq \|l\|_{L^{\infty}(\mathbb{R})} H_{\varphi}(\xi)$, we have again by the Poisson summation formula

$$\sum_{d=1}^{m} C_{\psi}(t_d) = \frac{1}{2\pi} \left\| \sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \prod_{d=1}^{m} e^{it_d(\xi+2n\pi)} \right\|^2_{L^2\ [0,2\pi]}$$

$$\leq \|l\|^2_{L^{\infty}(\mathbb{R})} \left\|H_{\varphi}(\xi)\right\|^2_{L^2\ [0,2\pi]}$$

so that $\sup_{\mathbb{R}} \sum_{d=1}^{m} C_{\psi}(t_d) < \infty$. For any $f \in L^2(\mathbb{R})$,

$$\sum_{d=1}^{m} \|L[f](t_d)\|_{L^2(\mathbb{R})} = \|f * l\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \left\|\hat{f}(\xi)\hat{l}(\xi)\right\|_{L^2(\mathbb{R})}$$

$$\leq \left\|\hat{l}\right\|_{L^{\infty}(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

Hence $L[\cdot] : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator so that for any

$\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$, $\sum_{d=1}^{m} L[f](t_d) = \sum_{d=1}^{m} (\boldsymbol{c} * \psi)(t_d)$ converges in $L^2(\mathbb{R})$. By (b), $\sum_{d=1}^{m} (\boldsymbol{c} * \psi)(t_d)$ also converges absolutely and uniformly on $\mathbb{R}$.

By Lemma 3.1(b), $\sum_{d=1}^{m} \psi(t_d) \in L^2(\mathbb{R})$. However, $\sum_{d=1}^{m} (\boldsymbol{c} * \psi)(t_d)$ may not converge in $L^2(\mathbb{R})$ unless $\{\sum_{d=1}^{m} \psi(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence.

Lemma 3.1(b) improves Lemma 1 in [9], in which the proof uses $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$,

$sup_{\mathbb{R}} \sum_{d=1}^{m} C_{\varphi}(t_d) < \infty$, and the integral version of Minkowski inequality. Note that the condition $H_{\varphi}(\xi) \in L^2[0, 2\pi]$ implies $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R}) \cap C_{\infty}((\mathbb{R})$ and $sup_{\mathbb{R}} \sum_{d=1}^{m} C_{\varphi}(t_d) < \infty$. (see [13]). Note also that $H_{\varphi}(\xi) \in L^2[0, 2\pi]$ if $\hat{\varphi}(\xi) = O\left((1 + |\xi|)^{-(1+\epsilon_2)}\right)$, $(1 + \epsilon_2)_{(1+\epsilon_1)} > 1, \epsilon_1 \geq 0$, which holds e.g. for $\sum_{d=1}^{m} \varphi_n(t_d) = \sum_{d=1}^{m} (\varphi_0 * \varphi_{n-1})(t_d)$ the cardinal B-spline of degree n (≥ 1), where

$\varphi_0 = \sum_{d=1}^{m} \chi_{[0,1)}(t_d)$. We have as a consequence of Lemma 3.1: Let $L[\cdot]$ be an LTI system with impulse response $\sum_{d=1}^{m} l(t_d)$ of type (i) or (ii) or (iii) as above and $\sum_{d=1}^{m}\psi(t_d) = \sum_{d=1}^{m}L[\varphi](t_d)$. Then for any $\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(\mathcal{J}F)(t_d) \in \sum_{d=1}^{m}V\left(\varphi(t_d)\right), F(\xi) \in L^2[0,2\pi]$

$$\sum_{d=1}^{m}L[f](t_d) = \sum_{d=1}^{m}\langle(\xi), \frac{1}{2\pi}\overline{Z_{\psi}(t_d,\xi)}\rangle_{L^2[0,2\pi]} \tag{2}$$

since $L[\cdot]$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ or $L^\infty(\mathbb{R})$ and $\{\sum_{d=1}^{m}\psi(t_d - n): n \in \mathbb{Z}\} \in l^2$, $\sum_{d=1}^{m}t_d \in \mathbb{R}$. Let $\sum_{d=1}^{m}\psi_{(1+\epsilon_1)}(t_d) = \sum_{d=1}^{m}L_{(1+\epsilon_1)}[\varphi](t_d)$ and

$g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi}Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)},\xi)$, $\epsilon_1 \geq 0$. Then we have by (2)

$$L_{(1+\epsilon_1)}[f]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}n\right)$$
$$= \langle F(\xi), \frac{1}{2\pi}Z_{\psi_{(1+\epsilon_1)}}\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}n, \xi\right)\rangle_{L^2[0,2\pi]}$$

$$= \langle F(\xi), \overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}\rangle_{L^2[0,2\pi]} \tag{3}$$

for any $\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(\mathcal{J}F)(t_d) \in \sum_{d=1}^{m}V(\varphi(t_d))$ and $\epsilon_1 \geq 0$. Then by (3) and the isomorphism $\mathcal{J}$ from $L^2[0,2\pi]$ onto $\sum_{d=1}^{m}V(\varphi(t_d))$, the sampling expansion (1) is equivalent to

$$F(\xi) = \sum_{\epsilon_1=0}^{N}\sum_{n\in\mathbb{Z}}\langle F(\xi), \overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}\rangle_{L^2[0,2\pi]}S_{(1+\epsilon_1),n}(\xi),$$

$F(\xi) \in L^2[0,2\pi]$, where $\{S_{(1+\epsilon_1),n}(\xi): \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $L^2[0,2\pi]$. This observation leads us to consider the problem when is $\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\}$ a series of frames or a Riesz basis of $L^2[0,2\pi]$. Note that

$$\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}: \epsilon_1 \geq 0, n \in \mathbb{Z}\right\} = \left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

where $(1+\epsilon_2) = lcm\{(1+\epsilon_2)_{(1+\epsilon_1)} : \epsilon_1 \geq 0\}$ and $g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) = g_{(1+\epsilon_1)}(\xi)e^{i(1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)}-1)\xi}$ for $\epsilon_1 \geq 0$.

Let D be the unitary operator from L^2 $[0,2\pi]$onto $L^2(I)^{(1+\epsilon_2)}$, where $I = [0,\frac{2\pi}{(1+\epsilon_2)}]$, defined by

$DF = \left[F\left(\xi + (k-1)\frac{2\pi}{(1+\epsilon_2)}\right)\right]_{k=1}^{(1+\epsilon_2)}, F(\xi) \in L^2$ $[0,2\pi]$. We also let

$G(\xi) =$

$$\left[Dg_{1,1}(\xi),\ldots,Dg_{1,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_1}}(\xi),\ldots,Dg_{N,1}(\xi),\ldots,Dg_{N,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_N}}(\xi)\right]^T \quad (4)$$

be a $\left(\sum_{\epsilon_1=0}^{N}\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right)$ $\times (1+\epsilon_2)$ matrix on I and $\lambda_m(\xi)$, $\lambda_M(\xi)$be the smallest and the largest eigenvalues of the positive semi-definite $(1+\epsilon_2)\times(1+\epsilon_2)$ matrix $G(\xi) * G(\xi)$, respectively.

Lemma 3.2. Let $\alpha_G = \|\lambda_m(\xi)\|_0$ and $\beta_G = \|\lambda_M(\xi)\|_\infty$ *be* the essential infimum of $\lambda_m(\xi)$ and the essential supremum of $\lambda_M(\xi)$ respectively. Then $\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}: \epsilon_1 \ge 0, n \in \mathbb{Z}\right\}$ is

(a) a Bessel sequence in L^2 $[0,2\pi]$ if and only if $\beta_G < \infty$ or equivalently

$\{Z_{\psi_{(1+\epsilon_1)}}(\sigma_{(1+\epsilon_1)},\xi): \epsilon_1 \ge 0\} \in L^\infty$ $[0,2\pi]$,

(b) a frame of L^2 $[0,2\pi]$if and only if $0 < \alpha_G \le \beta_G < \infty$,

(c) a Riesz basis of L^2 $[0,2\pi]$if and only if $0 < \alpha_G \le \beta_G < \infty$ and

$$\sum_{\epsilon_1=0}^{N}\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1.$$

Proof . Since $\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}: \epsilon_1 \ge 0, n \in \mathbb{Z}\right\}$ is a Bessel sequence or a series of frames or a Riesz basis of L^2 $[0,2\pi]$ if and only if

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \ge 0, 1 \le m_{(1+\epsilon_1)} \le \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

is a Bessel sequence or a series of frames or a Riesz basis of L^2 $[0,2\pi]$respectively, all of the conclusions follow from Lemma 3 in [9].

Note that in [9], the authors use the Fourier transform $\hat{f}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^{m} f(t_d) \prod_{d=1}^{m} e^{-2\pi i t_d \xi}\, dt_d$ so that they use L^2 $[0,2\pi]$ instead of L^2 $[0,2\pi]$.

Assume that $0 < \alpha_G \leq \beta_G < \infty$ so that $\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-\mathrm{i}(1+\epsilon_2)_{(1+\epsilon_1)} n\xi} : \epsilon_1 \geq 0, n \in \mathbb{Z}\right\}$ or equivalently

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2) n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

is a series of frames of L^2 $[0,2\pi]$. Then we can show easily (see in [9]) that

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2) n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$$

has a series of dual frames of the form $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)e^{-i(1+\epsilon_2) n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ with $S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi) \in L^\infty$ $[0,2\pi]$ for $\epsilon_1 \geq 0$ and $1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$ satisfying

$$\left[DS_{1,1}(\xi),\ldots,DS_{1,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_1}}(\xi),\ldots,DS_{N,1}(\xi),\ldots,DS_{N,\frac{(1+\epsilon_2)}{(1+\epsilon_2)_N}}(\xi)\right] = \frac{(1+\epsilon_2)}{2\pi}\left[G(\xi)^\dagger + B(\xi)\left(I - G(\xi)G(\xi)^\dagger\right)\right], \qquad (5)$$

where $G(\xi)^\dagger = [G(\xi)^* G(\xi)]^{-1} G(\xi)^*$ is the pseudo-inverse of $G(\xi), B(\xi)$ is any $(1+\epsilon_2) \times \sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$ matrix with entries in $L^\infty(I)$, and I is the

$\left(\sum_{\epsilon_1=0}^{N}\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right)\times\left(\sum_{\epsilon_1=0}^{N}\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right)$ identity matrix .

In particular, when we choose $B(\xi\ ) = 0$ in (5), we have the canonical a series of dual frames of the frames

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )}e^{-i(1+\epsilon_2)n\xi}:\epsilon_1\geq 0, 1\leq m_{(1+\epsilon_1)} \leq\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n\ \in\ \mathbb{Z}\ \right\}.$$

We are now ready to give the foolowing results (see [22]). We first discuss the sampling expansion (1), which is a series of frames expansion in $\sum_{d=1}^{m} V(\varphi(t_d))$.

Theorem 3.3: Let α_G and β_G be the same as in Lemma 3.2. Assume $\beta_G\ <\infty$. Then the following are all equivalent.

(a) There is a series of frames

$$\left\{\sum_{d=1}^{m} S_{\ (1+\epsilon_1),m_{(1+\epsilon_1)}}\ (t_d-(1+\epsilon_2)n\ ) : \epsilon_1\geq 0\ , 1\ \leq m_{(1+\epsilon_1)} \leq\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n\ \in\ \mathbb{Z}\ \right\}$$

of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N}\ \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}}\ \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} L_{(1+\epsilon_1)}\ [f]\big(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\big(m_{(1+\epsilon_1)}-1\big) + (1+\epsilon_2)n\ \big)s_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (t_d-(1+\epsilon_2)n)$$

$$,\sum_{d=1}^{m} f(t_d)\ \in\ \sum_{d=1}^{m} V(\varphi(t_d)). \qquad (6)$$

(b) There is a series of frames $\sum_{d=1}^{m} s_{(1+\epsilon_1),n}\ (t_d) : \epsilon_1\geq 0\ , n\ \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} L_{(1+\epsilon_1)}\ [f]\big(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}n\big)s_{(1+\epsilon_1),n}(t_d),$$

$$\sum_{d=1}^{m} f(t_d)\in\sum_{d=1}^{m} V(\varphi(t_d)). \qquad (7)$$

(c) $0 < \alpha_G$.

Proof: Assume $\beta_G < \infty$. Then by Lemma 3.2 $\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi} : \epsilon_1 \geq 0, n \in \mathbb{Z}\right\}$ is a Bessel sequence in $L^2\,[0,2\pi]$. First (a) implies (b) trivially. Assume (b). Applying the isomorphism $\mathcal{J}^{-1}$ to (7) gives by (3)

$$F(\xi) = \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_J(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi}\rangle_{L^2\,[0,2\pi]} S_{(1+\epsilon_1),n}(\xi),$$

$$F(\xi) \in L^2\,[0,2\pi],$$

where $\{\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ is a series of frames of $L^2\,[0,2\pi]$. Then the Bessel sequence $\left\{\overline{g_{(1+\epsilon_1)}(\xi)}e^{-i(1+\epsilon_2)_{(1+\epsilon_1)}n\xi} : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\right\}$ is in fact a series of dual frames of $\{\sum_{d=1}^{m} s_{(1+\epsilon_1),n}(t_d) : 0 \leq \epsilon_1 \leq N, n \in \mathbb{Z}\}$ (see [2]). Hence (c) must hold by Lemma 3.2. Finally assume (c). Then $0 < \alpha_G \leq \beta_G < \infty$ that $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi} : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ is a series of frames of $L^2\,[0,2\pi]$. Then we have a frame expansion on $L^2\,[0,2\pi]$

$$F(\xi) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \left(\langle F(\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)}e^{-i(1+\epsilon_2)n\xi}\rangle_{L^2\,[0,2\pi]} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)e^{-i(1+\epsilon_2)n\xi}\right), F(\xi) \in L^2\,[0,2\pi], \tag{8}$$

where $S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)$'$s$ are given by (5). Then the sampling expansion (6) comes from (8) by applying the isomorphism $\mathcal{J}$ since

$$\langle F(\xi), g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi)e^{-i(1+\epsilon_2)n\xi}\rangle_{L^2\,[0,2\pi]}$$
$$= \langle F(\xi), \frac{1}{2\pi}\overline{Z_{\psi_J}\left[\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)} - 1\right) + (1+\epsilon_2)n, \xi\right]}\rangle_{L^2\,[0,2\pi]}$$
$$= L_{(1+\epsilon_1)}[f]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)} - 1\right) + (1+\epsilon_2)n\right)$$

for $\sum_{d=1}^{m}(\mathcal{J}F)(t_d) = \sum_{d=1}^{m} f(t_d)$.

Note that when $0 < \alpha_G \leq \beta_G < \infty$, the sampling series (6) converges not only in $L^2(\mathbb{R})$ but also uniformly on any subset of $\mathbb{R}$, on which $\sum_{d=1}^{m} C_{\varphi}(t_d)$ is bounded. Moreover since $\alpha_G > 0$, the rank of $G(\xi)$ is $(1+\epsilon_2)$ a.e.so that $1 \leq \sum_{\epsilon_1=0}^{N} \frac{1}{(1+\epsilon_2)_{(1+\epsilon_1)}}$, which means that the total sampling rate $\sum_{\epsilon_1=0}^{N} \frac{1}{(1+\epsilon_2)_{(1+\epsilon_1)}}$ of the sampling expansion (6)must be at least 1,the Nyquist sampling rate for signals in $\sum_{d=1}^{m} V(\varphi(t_d))$. In the extreme case we have:

Theorem 3.4: Let α_G and β_G be the same as in Lemma 3.2. Then there is a series of Riesz bases

$\left\{\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) : \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} \left(L_{(1+\epsilon_1)}[f]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)} - 1\right) + (1+\epsilon_2)n\right) S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d)\right), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)) \qquad (9)$$

if and only if $0 < \alpha_G \leq \beta_G < \infty$ *and* $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$. In this case, we also have

(i) $\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n) : \epsilon_1 \geq 0,\quad 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}$, and $n \in \mathbb{Z}$

(ii) $L_{(1+\epsilon_1)}[s_{k,m_k}]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)} - 1\right) + (1+\epsilon_2)n\right) = \delta_{(1+\epsilon_1),k}\,\delta_{n,0}$

for $0 \leq \epsilon_1, k \leq N$ and $n \in \mathbb{Z}$.

Proof: Assume $0 < \alpha_G \leq \beta_G < \infty$ and $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$. Then by Lemma 3.2,

$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )}e^{-i(1+\epsilon_2)n\xi}\colon \epsilon_1 \geq 0, 1\ \leq m_{(1+\epsilon_1)}\ \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n\ \in\ \mathbb{Z}\right\}$ is a series of Riesz bases of L^2 $[0,2\pi]$. Then we have

$$F\ (\xi\ ) = \sum_{\epsilon_1=0}^{N}\ \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}}\ \sum_{n\,\in\mathbb{Z}} \langle F\ (\xi), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )}e^{-i(1+\epsilon_2)n\xi}\rangle_{L^2\ [0,2\pi]} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ )e^{-i(1+\epsilon_2)n\xi}$$

$, F\ (\xi\ )\ \in L^2\ [0,2\pi],$ (10)

where $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )e^{-i(1+\epsilon_2)n\xi}\ \colon \epsilon_1 \geq 0\ \ , 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n \in\ \mathbb{Z}\right\}$ is the dual of $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )}e^{-i(1+\epsilon_2)n\xi}\colon \epsilon_1 \geq 0\ \ , 1\ \leq m_{(1+\epsilon_1)}\ \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n\ \in\ \mathbb{Z}\right\}$. Applying the isomorphism $\mathcal{J}$ to (10) gives (9), where

$$\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d)\ =\ \mathcal{J}\ \left((\xi\ )e^{-i(1+\epsilon_2)n\xi}\ \right)$$

$$= \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d - (1+\epsilon_2)n)$$

and $\mathcal{J}\ (S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ \ ))\ =\ \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d)$. Conversely assume that the series of Riesz bases expansions (9) holds on $\sum_{d=1}^{m} V(\varphi(t_d))$. Applying the isomorphism $\mathcal{J}^{-1}$ to (9) gives

$$F(\xi\ ) = \sum_{\epsilon_1=0}^{N}\ \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}}\ \sum_{n\,\in\mathbb{Z}} (\langle F\ (\xi\ ), \overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}\ (\xi\ )}e^{-i(1+\epsilon_2)n\xi}\rangle_{L^2\ [0,2\pi]}\mathcal{J}^{-1} \left(\ S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}\right)(\xi\ )),\qquad F\ (\xi\ )\ \in L^2\ [0,2\pi]$$

which is a series of Riesz bases expansions on L^2 $[0,2\pi]$. Then $\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ )}e^{-i(1+\epsilon_2)n\xi}\colon \epsilon_1 \geq 0, 1\ \leq m_{(1+\epsilon_1)}\ \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\ , n\ \in\ \mathbb{Z}\right\}$ must be a series of Riesz bases of L^2 $[0,2\pi]$ so that 0 $< \alpha_G \leq \beta_G < \infty$ and $\sum_{\epsilon_1=0}^{N} \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}} = 1$ by Lemma 3.2. As the a series of the dual Riesz bases of

$$\left\{\overline{g_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ )}e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0, 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\},$$

$\left\{\sum_{d=1}^{m} \mathcal{J}^{-1}\left(S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}\right)(t_d\ ): \epsilon_1 \geq 0\ , 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$ must be of the form $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ )e^{-i(1+\epsilon_2)n\xi}: \epsilon_1 \geq 0\ , 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}, n \in \mathbb{Z}\right\}$, where $\left\{S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ ): \epsilon_1 \geq 0\ , 1 \leq m_{(1+\epsilon_1)} \leq \frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}\right\}$ satisfy (5) with $B(\xi) = 0$. Hence

$$\sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)},n}(t_d) = \mathcal{J}\left(S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(\xi\ )e^{-i(1+\epsilon_2)n\xi}\right)$$

$$= \sum_{d=1}^{m} S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d-(1+\epsilon_2)n), \epsilon_1 \geq 0\ , n \in \mathbb{Z}.$$

Finally, we have

$$\sum_{d=1}^{m} s_{k,mk}(t_d) = \sum_{\epsilon_1=0}^{N} \sum_{m_{(1+\epsilon_1)}=1}^{\frac{(1+\epsilon_2)}{(1+\epsilon_2)_{(1+\epsilon_1)}}} \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} (L_{(1+\epsilon_1)}[s_{k,m_k}](\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}(m_{(1+\epsilon_1)}-1) + (1+\epsilon_2)n)\, S_{(1+\epsilon_1),m_{(1+\epsilon_1)}}(t_d-(1+\epsilon_2)n))$$

so that $L_{(1+\epsilon_1)}\left[s_{k,m_k}\right]\left(\sigma_{(1+\epsilon_1)} + (1+\epsilon_2)_{(1+\epsilon_1)}\left(m_{(1+\epsilon_1)}-1\right) + (1+\epsilon_2)n\right) = \delta_{(1+\epsilon_1)_1,k}\,\delta_{n,0}$.

When $N = 1$, write $L_1[\cdot], \sum_{d=1}^{m} l_1(t_d)$, σ_1, $(1 + \epsilon_2)_1$, and $\sum_{d=1}^{m} \psi_1(t_d)$ as $L[\cdot]$, $\sum_{d=1}^{m} l(t_d)$, σ, $(1 + \epsilon_2)$, and $\sum_{d=1}^{m} \psi(t_d)$

Corollary 3.5: (Cf. Theorem 3.1 in [11].) Let $N = 1$. Then there is a series of Riesz bases $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ such that

$$\sum_{d=1}^{m} f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} L[f](\sigma + (1 + \epsilon_2)n) s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)) \qquad (11)$$

if and only if $\epsilon_2 = 0$ and

$$0 < \left\| Z_\psi(\sigma, \xi) \right\|_0 \leq \left\| Z_\psi(\sigma, \xi) \right\|_\infty . \qquad (12)$$

In this case, we also have

(i) $\sum_{d=1}^{m} s_n(t_d) = \sum_{d=1}^{m} s(t_d - n), n \in \mathbb{Z}$

(ii) $\hat{s}(\xi) = \frac{\hat{\varphi}(\xi)}{Z_\psi(\sigma,\xi)}$

(iii) $L[s](\sigma + n) = \delta_{n,0}, n \in \mathbb{Z}$. (13)

Proof: Note that for $\epsilon_2 = 0$, $G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma, \xi)$ and $\lambda_m(\xi) = \lambda_M(\xi) = \left(\frac{1}{2\pi}\right)^2 \left| Z_\psi(\sigma, \xi) \right|^2$ so that $0 < \alpha_G \leq \beta_G < \infty$ if and only if (12) holds. Therefore, everything except (13) follows from Theorem 3.4. Finally applying (11) to $\sum_{d=1}^{m} \varphi(t_d)$ gives

$$\sum_{d=1}^{m} \varphi(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} \psi(\sigma + n) s(t_d - n)$$

from which we have (13) by taking the Fourier transform. When $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d)$ so that $L[\cdot]$ is the identity operator, Corollary 3.5 reduces to a series of regular shifted sampling on $\sum_{d=1}^{m} V(\varphi(t_d))$ (see Theorem 3.3 in [13]).

Remark 3.6: In (1), we may allow rational sampling periods. If $(1 + \epsilon_2)_{(1+\epsilon_1)} = \frac{p_{(1+\epsilon_1)}}{q_{(1+\epsilon_1)}}$, where $p_{(1+\epsilon_1)}$ and $q_{(1+\epsilon_1)}$ are coprime positive integers, then

$$\left[L_{(1+\epsilon_1)} [f]\left(\sigma_{(1+\epsilon_1)} + (1 + \epsilon_2)_{(1+\epsilon_1)} n\right) : n \in \mathbb{Z}\right] =$$
$$\left[L_{(1+\epsilon_1)} [f]\left(\sigma_{(1+\epsilon_1)} + (1 + \epsilon_2)_{(1+\epsilon_1)} (k - 1) + p_{(1+\epsilon_1)} n\right) : 1 \leq k \leq q_{(1+\epsilon_1)}, n \in \mathbb{Z}\right].$$

Hence the case of rational sampling periods $\{(1 + \epsilon_2)_{(1+\epsilon_1)}\}_{\epsilon_1=0}^{N}$ can be reduced to the case of integer sampling periods $\{p_{(1+\epsilon_1)}\}_{\epsilon_1=0}^{N}$=1 by extending the number of LTI systems involved. For example when $N = 1$, we have:

Corollary 3.7: Let $N = 1$ *and* $q \geq 2$ be an integer. Assume $Z_\psi(\sigma_{(1+\epsilon_1)}, \xi) \in L^\infty[0, 2\pi]$, $0 \leq \epsilon_1 \leq q-1$,

where $\sigma_{(1+\epsilon_1)} = \sigma + \frac{1}{q-1}(\epsilon_1)$. Then the following are all equivalent.

(a) There is a series of frames $\{\sum_{d=1}^m s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ for which $\sum_{d=1}^m f(t_d) = \sum_{n \in \mathbb{Z}} \sum_{d=1}^m L[f]\left(\sigma + \frac{1}{q-1}n\right) s_n(t_d)$, $\sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d))$.

(b) There is a series of frames $\left\{\sum_{d=1}^m s_{(1+\epsilon_1)}(t_d - n) : 0 \leq \epsilon_1 \leq q-1, n \in \mathbb{Z}\right\}$ of $\sum_{d=1}^m V(\varphi(t_d))$ for which

$$\sum_{d=1}^m f(t_d) = \sum_{d=1}^m \sum_{\epsilon_1=0}^{q-1} \sum_{n \in \mathbb{Z}} L[f]\big(\sigma_{(1+\epsilon_1)} + n\big) s_{(1+\epsilon_1)}(t_d - n), \quad \sum_{d=1}^m f(t_d) \in \sum_{d=1}^m V(\varphi(t_d)).$$

(c) $\left\|\sum_{\epsilon_1=0}^{q} |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)|\right\|_0 > 0$.

Proof : Since $\left\{L[f]\left(\sigma + \frac{1}{q-1}n\right) : n \in \mathbb{Z}\right\} = \{L[f](\sigma_{(1+\epsilon_1)} + n) : 0 \leq \epsilon_1 \leq q-1, n \in \mathbb{Z}\}$, we have a series of shifted symmetric multi-channel sampling for q LTI systems $\{L_{(1+\epsilon_1)}[\cdot] : 0 \leq \epsilon_1 \leq q-1\}$ with $L_{(1+\epsilon_1)}[\cdot] = L[\cdot], 0 \leq \epsilon_1 \leq q-1$. Then $g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi\big(\sigma_{(1+\epsilon_1)}, \xi\big), 0 \leq \epsilon_1 \leq q-1$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)|^2$. Hence $\alpha_G > 0$ if and only if $\left\|\sum_{\epsilon_1=0}^{q-1} |Z_\psi(\sigma_{(1+\epsilon_1)}, \xi)|\right\|_0 > 0$. Therefore, Corollary 3.7 is a consequence of Theorem 3.3.

4. Example

Let $\varphi_0 = \sum_{d=1}^m \chi_{[0,1)}(t_d)$ be the Haar scaling function and

$$\sum_{d=1}^{m} \varphi_1(t_d) = \sum_{d=1}^{m} (\varphi_0 * \varphi_0)(t_d)$$
$$= \sum_{d=1}^{m} \chi_{[0,1)}(t_d) + \sum_{d=1}^{m} (2 - t_d)\chi_{[1,2)}(t_d)$$

a B-spline of degree 1. Then $\sum_{d=1}^{m} \varphi_1(t_d)$ is a continuous series of Riesz generators [3], [22] and $sup_{\mathbb{R}} \sum_{d=1}^{m} C_{\varphi_1}(t_d) = sup_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \sum_{d=1}^{m} |\varphi_1(t_d + n)|^2 < \infty$. First we take

$N = 2, \sigma_1 = \sigma_2 = 0, (1 + \epsilon_2)_1 = 1, (1 + \epsilon_2)_2 = 2,$ and two LTI systems $L^1[\cdot]$ and $L^2[\cdot]$ with impulse responses $\sum_{d=1}^{m} l_1(t_d) = \sum_{d=1}^{m} \chi_{\left[\frac{-1}{2},0\right)}(t_d)$ and $\sum_{d=1}^{m} l_2(t_d) = \sum_{d=1}^{m} \chi_{[-1,\frac{-1}{2})}(t_d)$. Then it's easy to see that

$$g_1(\xi) = \frac{1}{2\pi} Z_{\psi_1}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_1(n) e^{-in\xi} = \frac{1}{16\pi}\left(1 + 3e^{-i\xi}\right),$$
$$g_2(\xi) = \frac{1}{2\pi} Z_{\psi_2}(0, \xi) = \frac{1}{2\pi} \sum_{n \in \mathbb{R}} \psi_2(n) e^{-in\xi} = \frac{1}{16\pi}\left(1 + 3e^{-i\xi}\right),$$

where $\sum_{d=1}^{m} \psi_{(1+\epsilon_1)}(t_d) = \sum_{d=1}^{m} L_{(1+\epsilon_1)}[\varphi](t_d)$. Hence

$$g_{1,1}(\xi) = g_1(\xi), g_{1,2}(\xi) = g_1(\xi) e^{i\xi}, g_{2,1}(\xi) = g_2(\xi)$$

so that (see (4))

$$G(\xi) = \left[D g_{1,1}, D g_{1,2}, D g_{2,1}\right]^T = \frac{1}{16\pi} \begin{bmatrix} 1 + 3e^{-i\xi} & 1 - 3e^{-i\xi} \\ 3 + e^{i\xi} & 3 - e^{i\xi} \\ 3 + e^{-i\xi} & 3 - e^{-i\xi} \end{bmatrix}$$

And $G(\xi)^* G(\xi) = \frac{1}{(16\pi)^2} \begin{bmatrix} 30 + 18\cos\xi & 8 + 6i\sin\xi \\ 8 - 6i\sin\xi & 30 + 18\cos\xi \end{bmatrix}$.

The eigenvalues of $G(\xi)^* G(\xi)$ are

$\frac{1}{(16\pi)^2}[30 + 18\cos\xi \pm \sqrt{100 - 36\cos^2\xi}]$ so that

$\frac{1}{(16\pi)^2} \leq \alpha_G = \|\lambda_m(\xi)\|_0 < \beta_G = \|\lambda_M(\xi)\|_\infty \leq \frac{58}{(16\pi)^2}$. Hence by Theorem 3.3, there is a series of frames $\{\sum_{d=1}^{m} s_{(1+\epsilon_1)}(t_d - 2n) : \epsilon_1 = 0,1,2,3 \ and\, n \in \mathbb{Z}\}$ of the space of linear splines $\sum_{d=1}^{m} V(\varphi_1(t_d))$ for which the following series of asymmetric multi-channel sampling expansions holds:

$$\sum_{d=1}^{m} f(t_d) =$$

$$\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\{L_1\,[f](2n)s_1(t_d-2n)+L_1[f](2n+1)s_2(t_d\;-2n)$$

$$+\;L_2[f](2n)s_3(t_d-2n)\}\;,f\;\in\sum_{d=1}^{m}V\;(\varphi_1(t_d)),$$

which converges in $L^2(\mathbb{R})$ and absolutely and uniformly on $\mathbb{R}$.

We now take $N = 1$ and $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m}\delta(t_d)$ so that $L[\cdot]$ is the identity operator. Let $q\;(\geq 1)$ be an integer and $0 \leq \sigma < \frac{1}{q}$. Note first that for any fixed

$$\sum_{d=1}^{m} t_d \text{ in } \mathbb{R}\,, \sum_{d=1}^{m} Z_{\varphi_1}(t_d,\xi) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\varphi_1(t_d\;+\;n)e^{-in\xi}\;\in\;C[0{,}2\pi]$$

since $\sum_{d=1}^{m}\varphi_1(t_d)$ has compact support.Hence $\sum_{d=1}^{m}\left\|Z_{\varphi_1}(t_d,\cdot)\right\|_{L^\infty[0,2\pi]} < \infty$ for each $\sum_{d=1}^{m} t_d$ in $\mathbb{R}$. Since $Z_{\varphi_1}(\sigma,\xi) = \sigma + (1-\sigma)e^{-i\xi}$ for $0 \leq \sigma < 1$, $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_0 = 2|\sigma - \frac{1}{2}|$ and $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_\infty = 1$. Therefore, by Corollary 3.5, for any σ with $0 \leq \sigma < 1$, there is a series of Riesz bases $\{\sum_{d=1}^{m} s(t_d - n) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi\,(t_d))$ such that

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} f(\sigma+n)s(t_d-n)\quad,\sum_{d=1}^{m} f(t_d)\in\sum_{d=1}^{m}V(\varphi_1(t_d))$$

if and only if $\sigma \neq \frac{1}{2}$. On the other hand, by Corollary 3.7, for any $q\,(\geq 2)$ and any σ with $0 \leq \sigma < \frac{1}{q}$, there is a series of frames $\{\sum_{d=1}^{m} s_{(1+\epsilon_1)}\big(t_d - n\big) : 0 \leq \epsilon_1 \leq q-1, n \in \mathbb{Z}\}$ such that

$$\sum_{d=1}^{m} f(t_d) = \sum_{\epsilon_1=0}^{q-1}\sum_{\in\mathbb{Z}}\sum_{d=1}^{m} f\Big(\sigma + \frac{1}{q-1}(\epsilon_1)$$

$$+\,n\Big)s_{(1+\epsilon_1)}\,(t_d - n)\;,\sum_{d=1}^{m} f(t_d)\;\in\sum_{d=1}^{m}V(\varphi_1(t_d)).$$

As an example we show the following Corollary (see [22])

Corollary 3.7: Assume $Z_\psi(2-\epsilon\,,\xi)\in L^\infty[0,2\pi], 0\leq\epsilon_1\leq q-1$, then the following are all equivalent.

(a) There is a series of frames $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L[f](2-\epsilon)s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(b) There is a series of frames $\{\sum_{d=1}^{m} s_{(1+\epsilon_1)}(t_d - n) : \epsilon_1 > 0, n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{\epsilon_1\geq 0} \sum_{d=1}^{m} L[f](n-\epsilon)s_{(1+\epsilon_1)}(t_d - n)$, $\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$.

(c) $\left\| \sum_{\epsilon_1\geq 0} |Z_\psi(2-\epsilon, \xi)| \right\|_0 > 0$.

Proof: Since

$\{L[f](2-\epsilon)\} = \{L[f](n-\epsilon) : n \in \mathbb{Z}\}$. Now we have $\{L_{(1+\epsilon_1)}[\cdot] : \epsilon_1 > 0\}$ with

$L_{(1+\epsilon_1)}[\cdot] = L[\cdot], \epsilon_1 > 0$. Then $g_{(1+\epsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi(2-\epsilon, \xi), \epsilon_1 > 0$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\epsilon_1\geq 0} \left| Z_\psi(2-\epsilon, \xi) \right|^2$. Therefor $\alpha_G > 0$ if and only if

$\left\| \sum_{\epsilon_1\geq 0} |Z_\psi(2-\epsilon, \xi)| \right\|_0 > 0$.

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