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Sharp bounds for weakly singular integral inequalities On ordered fractional differential and integral equations

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Abstract:

Weakly singular integral inequalities of Gronwall–Bellman type are established, that generalized some known weakly singular inequalities which can be used in various problems in the theory of differential equations, integral equations and evolution equations. Applications to fractional differential and integral equations with sharp bounds are also shown.

Keywords: Integral inequality; Weakly singular; Fractional differential and integral equations

Introduction

The Gronwall type integral inequalities play a dominant role in the study of quantitative properties of solutions of differential and integral equations (see[1],[2],[3],[4]). The integrals concerning this type of inequalities have regular or continuous kernels, but some problems require us to solve integral inequalities with singular kernels. D. Henry [5] used this type of integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem; Sano and Kunimatsu [6] gave a sufficient condition for stabilization of

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semilinear parabolic distributed systems by making use of a modification of Henry's type inequality. Ye, Gao and Ding [7] also proved a generalized type of this inequality and used it to study the dependence of the solution on the order and the initial condition of a fractional differential equation. All this type inequalities are proved by an iteration argument and the estimation formulas are expressed by a complicated power series. Medveď [8] presented a new method to solve Henry's type inequalities and got the explicit bounds with a quite simple formulas which are similar to the classic Gronwall–Bellman inequalities. Qing-HuaMa and Josip Pečarić in [17] used the modification of Medveď's method to study a certain class of nonlinear inequalities of Henry's type, which generalizes some known results and can be used in the study of differential equations and integral equations. Some applications of the result to fractional differential and integral equations are also studied in [17].In addition, we use the method of [17] with specific changes in order and show certain estimates.

Main result

Let R denote the set of real numbers, $R_+ = [0, +\infty)$; $C^i(M, S)$ denotes the class of all i-times continuously differentiable defined on set M with range in the set S ($i = 1, 2, \ldots$) and $C^0(M, S) = C(M, S)$.

We cite some useful lemmas and definitions as follows:

Lemma 1: (See [9].) Let $a \geq 0$, $p = q + \epsilon \geq 0 \, and \, p \neq 0$, then set $\epsilon_1 = -\epsilon$

$$a^{1+\frac{\epsilon_1}{p}} \leq \left(1 + \frac{\epsilon_1}{p}\right) k^{\frac{\epsilon_1}{p}} - \frac{\epsilon_1}{p} k^{1+\frac{\epsilon_1}{p}} \leq \left(1 + \frac{\epsilon_1}{p}\right) k^{\frac{\epsilon_1}{p}} - K \frac{\epsilon_1}{p} \, k^{\frac{\epsilon_1}{p}} \leq k^{\frac{\epsilon_1}{p}}$$

For any $k > 0$.

Definition 2: (See [10].) Let $[x, y, z]$ be certain ordered parameter group of nonnegative real numbers. The group belongs to the first class distribution and is denoted by $[x, y, z] \in I$ if conditions $x \in (0, 1], y \in \left(\frac{1}{2}, 1\right)$ and $z \geq \frac{3}{2} - y$ are satisfied; The group belongs to the second class distribution and is denoted by $[x, y, z] \in II$ if conditions $x \in (0, 1], y \in (0, \frac{1}{2}]$ and $z > (1 - 2y^2)/(1 - y^2)$ are satisfied.

Lemma 3: (See [11]). Let $\alpha, 2\alpha, 3\alpha$ and p be certain ordered positive constants. Then

$$\int_0^t (t^{\alpha} - s^{\alpha})^{p(2\alpha-1)}\, s^{p(3\alpha-1)} ds = \frac{t^{\theta}}{\alpha} B\left[\frac{p(3\alpha-1)+1}{\alpha}, p(2\alpha-1)+1\right],$$

$$t \in R_+,$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1} (1-s)^{\eta-1} ds$ $(\Re\xi > 0,\ \Re\eta > 0)$ is the well-known B-function and $\theta = p[2\alpha(\alpha+1)-1]+1$.

Lemma 4: (See [10].) Suppose that the positive constants $\alpha, 2\alpha, 3\alpha, p_1$ and p_2 satisfy conditions:

$(a) if\ [\alpha, 2\alpha, 3\alpha] \in I\ , p_1 = \frac{1}{2\alpha}$;

$(b)\ if\ [\alpha, 2\alpha, 3\alpha] \in II, p_2 = \frac{1+8\alpha}{1+6\alpha},\ then$

$B\left[\frac{p_i(3\alpha-1)+1}{\alpha}, p_i(2\alpha-1)+1\right] \in (0, +\infty)$

and

$\theta_i = p_i[2\alpha(\alpha+1)-1]+1 \geq 0$

are valid for $i = 1{,}2$.

Lemma 5: (See [12].) Let $u(t), f(t), g(t)$ and $h(t)$ be nonnegative continuous functions on R_+, and let $r = 1 + \epsilon_2$ be a real number. If

$$u(t) \leq u_0(t) + w(t)\left[\int_0^t v(s) u^{1+\epsilon_2}(s) ds\right]^{1/(1+\epsilon_2)}, \qquad t \in R_+,$$

then

$$\int_0^t v(s) u^{1+\epsilon_2}(s) ds$$

$$\leq \left[1 - (1 - W(t)^{1/(1+\epsilon_2)}\right]^{-(1+\epsilon_2)} \int_0^t v(s) u_0^{1+\epsilon_2}(s) W(s) ds\,, t \in R_+,$$

where

$$W(t) = exp\left(-\int_0^t \mathrm{v(s)} w^{1+\epsilon_2} \mathrm{(s)ds}\right).$$

Theorem 6: Let $u(t), a(t), b(t)$ and $f(t)$ be nonnegative continuous functions for $t \in R_+$. Let $p = q + \epsilon \geq 0$. If $u(t)$ satisfies

$$u^p \leq a(t) + b(t) \int_0^t (t^\alpha - s^\alpha)^{(2\alpha-1)} s^{(3\alpha-1)} ds, \; t \in R_+, \tag{1}$$

then for any $K > 0$ we have

(i) if $[\alpha, 2\alpha, 3\alpha] \in I$,

$$u(t) \leq \Bigg\{ a(t) + M_1^{2\alpha} t^{2\alpha(\alpha+2)-1} b(t) \Bigg[\mathcal{A}_1^{1-2\alpha}(t) + K^{\frac{\epsilon_1}{p}} M_1^{\beta} \left[1 - \left(1 - V_1(t)\right)^{1-2\alpha} \right]^{-1} \times \left(\int_0^t s^{\frac{2\alpha(\alpha+2)-1}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) \mathcal{A}_1(s) V_1(s) ds \right)^{1-2\alpha} \Bigg] \Bigg\}^{\frac{1}{p}} \tag{2}$$

where

$$M_1 = \left[\frac{5\alpha-1}{\alpha^2}, \frac{4\alpha-1}{\alpha}\right], \qquad A(t) = \left(1 + \frac{\epsilon_1}{p}\right) K^{\frac{\epsilon_1}{p}} a(t) - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}},$$

$$\mathcal{A}_1(\mathrm{t}) = \int_0^1 f^{\frac{1}{1-2\alpha}}(\mathrm{s}) A^{\frac{1}{1-2\alpha}}(s) ds$$

and

$$V_1(t) = exp\left(-K^{\frac{-\epsilon_1}{p(1-2\alpha)}} M_1^{\frac{2\alpha}{1-2\alpha}} \int_0^t s^{\frac{2\alpha(\alpha+2)-1}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) \, b^{\frac{1}{1-2\alpha}}(s) ds \right);$$

(ii) if $[\alpha, 2\alpha, 3\alpha] \in II$,

$$\mathrm{u(t)} \leq \Bigg\{ \mathrm{a(t)} + \mathrm{M}_2^{\frac{1+6\alpha}{1+8\alpha}} \mathrm{t}^{\frac{2\alpha^2(8\alpha+9)}{1+16\alpha}} \mathrm{b(t)} \Bigg[\mathcal{A}_2^{\frac{2\alpha}{1+8\alpha}}(\mathrm{t}) + \mathrm{K}^{\frac{\epsilon_1}{\mathrm{p}}} \mathrm{M}_2^{\frac{1+6\alpha}{1+8\alpha}} \left[1 - \left(1 - \mathrm{V}_2(\mathrm{t})\right)^{\frac{2\alpha}{1+8\alpha}} \right]^{-1} \times \left(\int_0^t s^{\alpha(8\alpha+9)} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}} \mathcal{A}_2(s) \mathrm{V}_2(\mathrm{s}) \mathrm{ds} \right)^{\frac{2\alpha}{1+8\alpha}} \Bigg] \Bigg\}^{\frac{1}{\mathrm{p}}}, \tag{3}$$

where

$$M_2 = 2\left[\frac{1 + 24\alpha}{1 + 6\alpha}, \frac{16\alpha^2}{1 + 6\alpha}\right], \quad \mathcal{A}_2(t) = \int_0^t f^{\frac{1+8\alpha}{2\alpha}}(s) A^{\frac{1+8\alpha}{2\alpha}}(s) ds$$

and

$$V_2(t) = exp\left(-K^{\frac{\epsilon_1(1+8\alpha)}{2p\alpha}} M_2^{\frac{1+6\alpha}{2\alpha}} \int_0^t s^{\alpha(8\alpha+9)} f^{\frac{1+8\alpha}{2\alpha}}(s)\, b^{\frac{1+8\alpha}{2\alpha}}(s)ds\right).$$

Proof. Define a function $v(t)$ by

$$v(t) = b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{3\alpha-1} f(s) u^{p+\epsilon_1}(s)ds, \quad t \in R_+ \tag{4}$$

then

$$u^p(t) \leq a(t) + v(t)$$

or

$$u(t) \leq \left(a(t) + v(t)\right)^{\frac{1}{p}}. \tag{5}$$

By Lemma 1 and (5), for any $K > 0$, we have

$$u^{p+\epsilon_1}(t) \leq \left(a(t) + v(t)\right)^{1+\frac{\epsilon_1}{p}}$$
$$\leq \left(1 + \frac{\epsilon_1}{p}\right) \mathrm{K}^{\frac{\epsilon_1}{p}}\left(\mathrm{a(t) + v(t)}\right) - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}}.$$

Substituting the last relations into (4) we get

$$u(t) \leq b(t)\int_0^t (t^\alpha - s^\alpha)^{2\alpha-1} s^{3\alpha-1} f(s)ds\left[\left(1 + \frac{\epsilon_1}{p}\right)\mathrm{K}^{\frac{\epsilon_1}{p}}(\mathrm{a(s) + v(s)}) - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}}\right]ds$$
$$= b(t)\int_0^t (t^\alpha - s^\alpha)^{2\alpha-1} s^{3\alpha-1} f(s)A(s)ds$$
$$+ \left(1 + \frac{\epsilon_1}{p}\right)\mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{b(t)}\int_0^t (t^\alpha - s^\alpha)^{2\alpha-1} s^{3\alpha-1} f(s)v(s)ds, \tag{6}$$

where $A(t) = \left(1 + \frac{\epsilon_1}{p}\right)\mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{a(t)} - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}}$.

If $[\alpha, 2\alpha, 3\alpha] \in I$, let $p_1 = \frac{1}{2\alpha}$, $q_1 = \frac{1}{1-2\alpha}$; $if\ [\alpha, 2\alpha, 3\alpha] \in II$, let $p_2 = (1 + 8\alpha)/(1 + 6\alpha), q_2 = (1 + 8\alpha)/2\alpha$, then $\frac{1}{p_i} + \frac{1}{q_i} = 1\ for\ i = 1,2$,and then using Hölder's inequality with indexes p_i, q_i to (6) we get

$$u(t) \leq b(t)\left[\int_0^t (t^\alpha - s^\alpha)^{p_i(2\alpha-1)} s^{p_i(3\alpha-1)} ds\right]^{1/p_i}\left[\int_0^t f^{q_i}(s)A^{q_i}(s)ds\right]^{1/q_i}$$

$$+\mathrm{K}^{\frac{-\epsilon}{p}}\mathrm{b(t)}\left[\int_0^t (t^{\alpha}-s^{\alpha})^{p_i(2\alpha-1)}\, s^{p_i(3\alpha-1)}ds\right]^{1/p_i}\left[\int_0^t f^{q_i}(s)v^{q_i}(s)ds\right]^{1/q_i}.$$

By Lemmas 3 and 4, the last inequality can be rewritten as

$$v(t) \leq \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}\mathcal{A}_i^{\frac{1}{q_i}}(t)b(t) + \mathrm{K}^{\frac{\epsilon_1}{p}}\left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}b(t)\left[\int_0^t f^{q_i}(s)v^{q_i}(s)ds\right]^{\frac{1}{q_i}} \quad (7)$$

for t ∈ R_+, where

$$M_i = 2\left[\frac{p_i(3\alpha-1)+1}{\alpha}, p_i(2\alpha-1)+1\right], \quad \mathcal{A}_\mathrm{i}(\mathrm{t}) = \int_0^t f^{q_i}(s)A^{q_i}(s)ds$$

and θ_i is given as in Lemma 4 for $i = 1, 2$.

Using Lemma 5 to (7), we get

$$v(t) \leq \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}\mathcal{A}_i^{\frac{1}{q_i}}(t)b(t) + \mathrm{K}^{\frac{\epsilon_1}{p}}\left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}b(t)\left[1-(1-V_i(t))^{\frac{1}{q_i}}\right]^{-1}$$

$$\times\left(\int_0^1 f^{q_i}(s)\left(M_i t^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s)\mathcal{A}_i(s)V_i(s)ds\right)^{\frac{1}{q_i}}, \quad (8)$$

where

$$V_i(t) = exp\left(-\mathrm{K}^{\frac{q_i\epsilon_1}{\mathrm{p}}}\int_0^t f^{q_i}(s)\left(M_i s^{\theta_i}\right)^{\frac{q_i}{p_i}} b^{q_i}(s)ds\right).$$

Finally, substituting (8) into (5), considering two situations for $i = 1, 2$ and using parameters $\alpha, 2\alpha\ and\ 3\alpha$ to denote $p_i, q_i\ and\ \theta_i\ in$ (8), we can get the desired estimations (2) and (3), respectively.

Remark 7: (i) In (2) and (3), we not only have given some new bounds (see [17]) to a class of nonlinear weakly singular integral inequalities, but also note that the functions $a(t)\ and\ b(t)$ appearing in (2) and (3) are not required to satisfy the nondecreasing condition as some known results [7,8,10].

(ii) Using the generalized Bernoulli inequality [13] to (2) and (3), we can obtain some simpler formulas to the estimates of the solutions of (1) as follows (see [17]).

Theorem 8 . *Let* $u(t), a(t), b(t), f(t),\ p = q + \epsilon$ be defined as in Theorem 6, $u(t)$ satisfy (1). Then for any $K > 0$ we have

$(i)\ if\ [\alpha, 2\alpha, 3\alpha] \in I$,

$$\mathrm{u(t)} \leq \left\{\mathrm{a(t)} + \mathrm{M}_1^{2\alpha}\mathrm{t}^{\alpha(2\alpha+1)+2}\mathrm{b(t)}\left[\mathcal{A}_1^{1-2\alpha}(\mathrm{t}) + \mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{M}_1^{2\alpha}\frac{\mathrm{M}_1^{2\alpha}}{1-2\alpha}\mathrm{V}_1^{-1}(\mathrm{t}) \times \left(\int_0^t s^{\frac{\alpha(2\alpha+1)+2}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s)\mathcal{A}_1(s)\mathrm{V}_1(\mathrm{s})\mathrm{ds}\right)^{1-2\alpha}\right]\right\}^{\frac{1}{\mathrm{p}}}, \tag{9}$$

where $M_1, \mathcal{A}_1(t) and\ \mathrm{V}_1(t)$ are defined as in Theorem 6 for $t \in R_+$;

$(ii)\ if\ [\alpha, 2\alpha, 3\alpha] \in II$,

$$\mathrm{u(t)} \leq \left\{\mathrm{a(t)} + \mathrm{M}_2^{\frac{1+6\alpha}{1+4\alpha}}\mathrm{t}^{\frac{2\alpha^2(8\alpha+9)}{1+8\alpha}}\mathrm{b(t)}\left[\mathcal{A}_2^{\frac{2\alpha}{1+8\alpha}}(\mathrm{t}) + \mathrm{K}^{\frac{\epsilon_1}{p}}\mathrm{M}_2^{\frac{1+6\alpha}{1+8\alpha}} \times \left(\frac{1+8\alpha}{2\alpha}\right)\mathrm{V}_2^{-1}(\mathrm{t})\left(\int_0^t s^{\frac{2\alpha^2(8\alpha+9)}{1+8\alpha}} f^{\frac{1+8\alpha}{2\alpha}}(s)\mathcal{A}_2(s)\mathrm{V}_2(\mathrm{s})\mathrm{ds}\right)^{\frac{2\alpha}{1+8\alpha}}\right]\right\}^{\frac{1}{\mathrm{p}}}, \tag{10}$$

where $\mathrm{M}_2, \mathrm{A}_2(\mathrm{t})$ and $\mathrm{V}_2(\mathrm{t})$ are defined as in Theorem 6 for $t \in R_+$.

Proof. By the generalized Bernoulli inequality [13], we have

$(1 - V_i(t))^{\frac{1}{q_i}} < 1 - \frac{1}{q_i}V_i(t)$

or

$$\left[1 - (1 - V_i(t))^{\frac{1}{q_i}}\right]^{-1} < q_i\mathrm{V}_\mathrm{i}^{-1}(\mathrm{t})$$

for $i = 1, 2$, where $V_i(t)$ is defined as in Theorem 6. Substituting the last inequalities into (2) and (3) we can obtain (9) and (10) respectively.

Corollary 9. Let functions $u(t), a(t), b(t)$ and $f(t)$ be defined as in Theorem 6. Suppose that

$$u(t) \leq\ a(t) +\ b(t)\int_0^t (t - s)^{2\alpha-1} f(s)u(s)ds,\ t \in R_+. \tag{11}$$

Then we have

$(i)\ \ if\ \alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$,

$$u(t) \le a(t) + M_{11}^{2\alpha} t^{8\alpha-1} b(t) \left[\mathcal{A}_{11}^{1-2\alpha}(t) + \frac{M_{11}^{2\alpha}}{1-2\alpha} V_{11}^{-1}(t) \int_0^t s^{\frac{8\alpha-1}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s) \mathcal{A}_{11}(s) V_{11}(s) ds \right], \qquad (12)$$

where

$$M_{11} = B\left[1, \frac{4\alpha - 1}{2\alpha}\right], \qquad \mathcal{A}_{11}(t) = \int_0^t f^{\frac{1}{1-2\alpha}}(s) a^{\frac{1}{1-2\alpha}}(s)\, ds$$

and

$$V_{11}(t) = exp\left(-M_{11}^{\frac{2\alpha}{1-2\alpha}} \int_0^t s^{\frac{4\alpha-1}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s) ds \right)$$

for $t \in R_+$;

(ii) if $\alpha \in (0, \frac{1}{4}]$,

$$u(t) \le a(t) + M_{12}^{\frac{1+6\alpha}{1+8\alpha}} t^{8\alpha} b(t) \left[\mathcal{A}_{12}^{\frac{2\alpha}{1+8\alpha}}(t) + \frac{1+8\alpha}{2\alpha} M_{12}^{\frac{1+6\alpha}{1+8\alpha}} V_{12}^{-1}(t) \times \int_0^t s^{8\alpha} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}}(s) \mathcal{A}_{12}(s) V_{12}(s) ds \right], \qquad (13)$$

where

$$M_{12} = B\left[1, \frac{8\alpha^2}{1 + 6\alpha}\right], \qquad \mathcal{A}_{12}(t) = \int_0^t f^{\frac{1+8\alpha}{2\alpha}}(s) a^{\frac{1+8\alpha}{2\alpha}}(s)\, ds$$

and

$$V_{12}(t) = exp\left(-M_{12}^{\frac{1+6\alpha}{2\alpha}} \int_0^t s^{8\alpha} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}}(s) ds \right)$$

for $t \in R_+$;

Proof: (12) and (13) follow by letting $p = \alpha = 3\alpha = 1$ in Theorem 8 and by simple computation.

Remark 10: Inequality (11) has been studied in [7], but here we not only have given some new estimates which are not in complicated power series, but also eliminated the nondecreasing condition to function $b(t)$ (see[17]).

Let $p = 2, p = \alpha = 3\alpha = 1$, we can get the following interesting Henry–Ou-Iang type singular integral inequality. About Ou-Iang type inequalities and their applications we refer to [4].

Corollary 11. Let functions $u(t), a(t), b(t)$ and $f(t)$ be defined as in Theorem 6. Suppose that

$$u^2(t) \leq a(t) + b(t)\int_0^t (t-s)^{2\alpha-1} f(s)u(s)ds, \ t \in R_+. \qquad (14)$$

Then for any $K > 0$ we have

(i) if $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$,

$$u(t) \leq \left\{a(t) + M_{11}^{2\alpha} t^{4\alpha-1} b(t)\left[\tilde{\mathcal{A}}_{11}^{1-2\alpha}(t) K^{-\frac{1}{2}} \frac{M_{11}^{2\alpha}}{1-2\alpha} \widetilde{V}_{11}^{-1}(t) \times \int_0^t s^{\frac{4\alpha-1}{1-2\alpha}} f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s) \tilde{\mathcal{A}}_{11}(s) \widetilde{V}_{11}(s)ds\right]\right\}^{\frac{1}{2}}, \qquad (15)$$

where

$$\tilde{\mathcal{A}}_{11}(t) = \left(\frac{1}{2}K^{\frac{1}{2}}\right)^{\frac{1}{1-2\alpha}} \int_0^t f^{\frac{1}{1-2\alpha}}(s)\left(\frac{a(s)}{K} + 1\right)^{\frac{1}{1-2\alpha}} ds,$$

$$\widetilde{V}_{11}(t) = exp\left[-\left(\frac{M_{11}^{2\alpha}}{K^{\frac{1}{2}}}\right)^{\frac{1}{1-2\alpha}} \int_0^t s^{\frac{4\alpha-1}{1-2\alpha}}(s) f^{\frac{1}{1-2\alpha}}(s) b^{\frac{1}{1-2\alpha}}(s)\, ds\right]$$

and M_{11} is defined as in Corollary 9 for $t \in R_+$;

(ii) if $\alpha \in \left(0, \frac{1}{4}\right]$,

$$u(t) \leq \left\{a(t) + M_{12}^{\frac{1+6\alpha}{1+8\alpha}} t^{\frac{16\alpha^2}{1+8\alpha}} b(t)\left[\tilde{\mathcal{A}}_{12}^{\frac{2\alpha}{1+8\alpha}}(t) K^{-\frac{1}{2}} M_{12}^{\frac{1+6\alpha}{1+8\alpha}} \left(\frac{1+8\alpha}{2\alpha}\right) \widetilde{V}_{12}^{-1}(t) \times \left(\int_0^t s^{\frac{16\alpha^2}{1+8\alpha}} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}}(s) \tilde{\mathcal{A}}_{12}(s) \widetilde{V}_{12}(s)ds\right)^{\frac{2\alpha}{1+8\alpha}}\right]\right\}^{\frac{1}{2}} \qquad (16)$$

where

$$\tilde{\mathcal{A}}_{12}(t) = \left(\frac{1}{2}K^{\frac{1}{2}}\right)^{\frac{1+8\alpha}{2\alpha}} \int_0^t f^{\frac{1+8\alpha}{2\alpha}}(s)\left(\frac{a(s)}{K} + 1\right)^{\frac{1+8\alpha}{2\alpha}}(s)\, ds,$$

$$\widetilde{V}_{12}(t) = exp\left[-\left(\frac{M_{12}^{1+6\alpha}}{K^{\frac{1+8\alpha}{2}}}\right)^{\frac{1}{2\alpha}} \int_0^t s^{\frac{16\alpha^2}{1+2\alpha}} f^{\frac{1+8\alpha}{2\alpha}}(s) b^{\frac{1+8\alpha}{2\alpha}}(s)\, ds\right]$$

and M_{12} is defined as in Corollary 9 for t ∈ R_+;

Proof: Inequalities (15) and (16) follow by letting $p = 2, p = \alpha = 3\alpha = 1$ in Theorem 8 and by simple computation.

Applications:

We will indicate the usefulness of the results of [17] in the study of the boundedness of certain fractional differential equations with Riemann–Liouville (R–L) fractional operator and Erdélyi–Kober (E–K) operator.

Riemann–Liouville derivative and integral, and Erdélyi–Kober (E–K) operator are defined as below, respectively:

Definition 12: (See [14].) The fractional derivative of order $0 < \alpha < 1$ of a function $f(x) \in C(R_+, R)$ is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t) dt$$

provided that the right side is pointwise defined on R_+.

Definition 13. (See [14].) The fractional primitive of order $\alpha > 0$ of a function $f : R_+ \to R$ is given by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

provided the right side is pointwise defined on R_+.

Definition 14. (See [15,16].) The Erdélyi–Kober fractional integral of a continuous $f : R_+ \to R$ is defined by

$$I^\alpha f(x) = \frac{x^{-8\alpha^2}}{\Gamma(\alpha)} \int_0^x (x^{2\alpha} - t^{2\alpha})^{\alpha-1} t^{6\alpha^2} f(t)\, d(t^{2\alpha})$$

with real $\alpha > 0$, provided the right side is pointwise defined on R_+.

(I)Consider the following initial value problem of Podlubny [14] in terms of the Riemann–Liouville fractional derivatives:

$$D^\alpha y(t) = f(t, y(t)), \tag{17}$$

$$D^{\alpha-1}y(t)|_{t=0} = \eta, \tag{18}$$

where $0 < \alpha < 1$, $0 \leq t < T \leq +\infty, f : [0, T) \times R \to R$; and D^{α} denotes **R–L** derivative operator.

From the problem (17)–(18) we can get a fractional integral equation

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, y(\tau)\big)\, d\tau, \tag{19}$$

which is equivalent to the initial value problem (17)–(18) (cf. [14]).

Theorem 15: Let $0 < \alpha \leq 1$ *and* f be continuous and satisfy the condition

$$|f(t,y)| \leq \mathrm{g}(t)|y|^{p+\epsilon_1}, \tag{20}$$

where $0 < p + \epsilon_1 \leq 1$ is a constant, g(t) is nonnegative continuous function for $0 \leq t < T \leq +\infty$. Then for any solutions $y(t)$ of the initial value problem (17)–(18)

(i) *if* $\alpha \in \left(\frac{1}{2}, 1\right)$,

$$|y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{11}^{\alpha}\, t^{2\alpha-1}}{\Gamma(\alpha)} \left[\mathcal{A}_{1(\mathrm{p}+\epsilon_1)}^{1-\alpha}(t) + \frac{K^{p+\epsilon_1-1}\widetilde{M}_{11}^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \widetilde{\mathrm{V}}_{1(\mathrm{p}+\epsilon_1)}^{-1}(\mathrm{t}) \times \left(\int_0^t s^{\frac{2\alpha-1}{1-\alpha}} \mathrm{g}^{\frac{1}{1-\alpha}}(s)\, \mathcal{A}_{1(\mathrm{p}+\epsilon_1)}(s) \widetilde{\mathrm{V}}_{1(\mathrm{p}+\epsilon_1)}(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], 0 < t < T \leq +\infty, \tag{21}$$

where

$$A_{p+\epsilon_1}(t) = \frac{(\mathrm{p}+\epsilon_1)|\eta|}{K^{1-(p+\epsilon_1)}\ \Gamma(\alpha)} t^{\alpha-1} + \big(1 - (p+\epsilon_1)\big)K^{\mathrm{p}+\epsilon_1},$$

$$\widetilde{M}_{11} = B\left[1, \frac{2\alpha-1}{\alpha}\right],$$

$$\mathcal{A}_{1(\mathrm{p}+\epsilon_1)}(t) = \int_0^1 \mathrm{g}^{\frac{1}{1-\alpha}}(s) A_{\mathrm{p}+\epsilon_1}^{\frac{1}{1-\alpha}}(s) ds$$

and

$$\widetilde{\mathrm{V}}_{1(\mathrm{p}+\epsilon_1)}(t) = exp\left[-\left(\frac{K^{1-(\mathrm{p}+\epsilon_1)}\ \widetilde{M}_{11}^{\alpha}}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}} \int_0^t s^{\frac{2\alpha-1}{1-\alpha}} \mathrm{g}^{\frac{1}{1-\alpha}}(s)\, ds\right];$$

(ii) *if* $\alpha \in (0, \frac{1}{2}]$,

$$|y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}$$

$$+ \frac{\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}} t^{4\alpha}}{\Gamma(\alpha)} \left[\mathcal{A}_{2(\mathrm{p}+\epsilon_1)}^{\frac{\alpha}{1+4\alpha}}(t) \right.$$

$$+ \frac{K^{\mathrm{p}+\epsilon_1-1} \widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}} (1+4\alpha)}{\alpha\Gamma(\alpha)} \widetilde{\mathrm{V}}_{2(\mathrm{p}+\epsilon_1)}^{-1}(\mathrm{t})$$

$$\left. \times \left(\int_0^t s^{A\alpha} \mathrm{g}^{\frac{1+4\alpha}{\alpha}}(s)\, \mathcal{A}_{2(\mathrm{p}+\epsilon_1)}(s) \widetilde{\mathrm{V}}_{2(\mathrm{p}+\epsilon_1)}(s) ds \right)^{\frac{\alpha}{1+4\alpha}} \right], \quad 0 < t < T \leq +\infty, \qquad (22)$$

where

$$\widetilde{M}_{12} = B\left[1, \frac{4\alpha^2}{1+3\alpha}\right],$$

$$\mathcal{A}_{2(\mathrm{p}+\epsilon_1)}(t) = \int_0^t \mathrm{g}^{\frac{1+4\alpha}{\alpha}}(s) A_{\mathrm{p}+\epsilon_1}^{\frac{1+4\alpha}{\alpha}}(s) ds$$

and

$$\widetilde{\mathrm{V}}_{2(\mathrm{p}+\epsilon_1)}(t) = exp\left[-\left(\frac{K^{p+\epsilon_1}}{\Gamma(\alpha)} \right)^{\frac{1+3\alpha}{\alpha}} \widetilde{M}_{12}^{\frac{1+3\alpha}{\alpha}} \int_0^t s^{4\alpha} \mathrm{g}^{\frac{1+4\alpha}{\alpha}}(s)\, ds \right].$$

Proof: From (19) and (20) we have

$$|y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left| f(\tau, y(\tau)) \right| d\tau$$

$$\leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \mathrm{g}(\tau) |y(\tau)|^{p+\epsilon_1} d\tau.$$

An application of Theorem 8 $(with\ a(t) = \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}, b(t) = \frac{1}{\Gamma(\alpha)}, f\ (t) = \mathrm{g}(t), p = 1, \alpha = 3, \alpha = 1\ and\ \alpha = 0)$ to the last inequality yields the desired estimations (21) and (22).

(II) Consider the following Volterra type integral equations of second kind, involving an E–K fractional integral with parameters $\alpha, 3\alpha\ and\ 2\alpha$,

$$y^p(t) - \lambda t^{-6\alpha^2} \int_0^t \frac{\left(t^{2\alpha}-\tau^{2\alpha}\right)^{\alpha-1}}{\Gamma(\alpha)} \tau^{2\alpha(\alpha+1)-1} y^{p+\epsilon_1}(\tau) d(\tau) = f(t), \quad (23)$$

which arises very often in various problems. When (23) is a linear equation, i.e., $p = 1$, the other parameters satisfy some conditions and $y(t)$ belong to a space of weighted continuous functions, Al-Saqabi and Kiryakova [16] have found the solutions of (23) in the explicit form with convolutional type integral involving Mittag–Leffler function. Here we give the explicit bound of the solutions of nonlinear equation (23) under some suitable conditions (see[17]).

Theorem 16: Let $y(t), f(t) \in C[0, +\infty)$, $p = q + \epsilon > 0$ be constants and $y(t)$ satisfy (23). Then for any constant $K > 0$ we have

$(i)\ if\ [\alpha, 2\alpha, 2\alpha(1 + 3\alpha)] \in I$,

$$|y(t)| \le \left\{ |f(t)| + \frac{|\lambda| \bar{M}_1^{\alpha}}{\Gamma(\alpha)} t^{\alpha(2\alpha+1)-1} \left[\bar{\mathcal{A}}_1^{1-\alpha}(t) + K^{\frac{\epsilon_1}{p}} \frac{|\lambda| \bar{M}_1^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \bar{\mathrm{V}}_1^{-1}(\mathrm{t}) \right.\right.$$

$$\left.\left. \times \left(\int_0^t s^{\frac{\alpha(2\alpha+1)-1}{1-\alpha}}(s)\, \bar{\mathcal{A}}_1(s) \bar{V}_1(s) ds \right)^{1-\alpha} \right] \right\}^{\frac{1}{p}} 0 > t, \quad (24)$$

where

$$\bar{M}_1 = \frac{1}{2\alpha} B \left[\frac{3\alpha(2\alpha + 1) - 1}{2\alpha^2}, \frac{2\alpha^2 - 1}{\alpha} \right],$$

$$\bar{A}(t) = \left(1 + \frac{\epsilon_1}{p}\right) K^{\frac{\epsilon_1}{p}} |f(t)| - \frac{\epsilon_1}{p} K^{1+\frac{\epsilon_1}{p}},$$

$$\bar{\mathcal{A}}_1(t) \int_0^t \bar{\mathcal{A}}^{\frac{1}{1-\alpha}}(s) ds$$

and

$$\bar{V}_1(t) = exp \left[-\frac{(1-\alpha) K^{\frac{-\epsilon_1}{p(1-\alpha)}}}{2\alpha^2} \left(\frac{\bar{M}_1^{\alpha} |\lambda|}{\Gamma(\alpha)} \right)^{\frac{1}{1-\alpha}} t^{\frac{2\alpha^2}{1-\alpha}} \right];$$

$(ii)\ if\ [\alpha, 2\alpha, 2\alpha(1 + 3\alpha)] \in II$,

$$|y(t)| \le$$

$$\left\{ |f(t)| + \frac{|\lambda| \bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}}}{\Gamma(\alpha)} t^{4\alpha(5\delta+2)-1} \left[\bar{\mathcal{A}}_2^{\frac{\alpha}{1+4\alpha}}(t) + \frac{K^{\frac{\epsilon_1}{p}} \bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}} (1+4\alpha)|\lambda|}{\alpha\Gamma(\alpha)} \right.\right.$$

$$\left.\left. \times \bar{\mathrm{V}}_2^{-1}(\mathrm{t}) \left(\int_0^t s^{2\alpha(4\delta+1)-1}(s)\, \bar{\mathcal{A}}_2(s) \bar{V}_2(s) ds \right)^{\frac{\alpha}{1+4\alpha}} \right] \right\}^{\frac{1}{p}} 0 > t, \quad (25)$$

where

$$\bar{M}_2 = \frac{1}{2\alpha} B\left[\frac{2\alpha(12\alpha+7)+1}{2(1+4\alpha)}, \frac{4\alpha^2}{1+3\alpha^2}\right], \qquad \bar{\mathcal{A}}_2(t) \int_0^t \bar{\mathcal{A}}^{\frac{1+4\alpha}{\alpha}}(s)ds$$

and

$\bar{V}_2(t)$

$$= exp\left[-\frac{K^{\frac{\epsilon_1(1+4\alpha)}{p\alpha}} \bar{M}_2^{\frac{1+3\alpha}{\alpha}}}{2\alpha(1+4\alpha)}\left(\frac{|\lambda|}{\Gamma(\alpha)}\right)^{\frac{1+4\alpha}{\alpha}} t^{\,2\alpha(1+4\alpha)}\right].$$

Proof. From (23) we have

$|y|^p(t)$

$$\leq |f(t)| + \frac{|\lambda|}{\Gamma(\alpha)} t^{-6\alpha^2} \int_0^t (t^{2\alpha} - \tau^{2\alpha})^{\alpha-1} \tau^{2\alpha(3\alpha+1)-1} |y|^{p+\epsilon_1}(\tau)d(\tau).$$

An application of Theorem 8 $(with\, a(t) = f(t), b(t) = \frac{|\lambda|}{\Gamma(\alpha)} t^{-6\alpha^2}, \alpha = 0$ and $\alpha = \frac{1}{6}$ to the last inequality yields the desired estimations (24) and (25).

Remark 17: Obviously, the boundedness of the solutions of (17)–(18) and (23) cannot be derived by the known results in [5–8,10].

Letting $p = 1$ in Theorem 16, we can obtain an interesting result as follows.

Corollary 18. Let $y(t), f\,(t) \in C[0, +\infty)$ and $y(t)$ satisfy the equation

$$y(t) - \lambda t^{-6\alpha^2} \int_0^t \frac{\left(t^{2\alpha} - \tau^{2\alpha}\right)^{\alpha-1}}{\Gamma(\alpha)} \tau^{2\alpha(3\alpha+1)-1} y(\tau)d(\tau) = f(t), \qquad (26)$$

Then we have

$(i)\; if\; [\alpha, 2\alpha, 2\alpha(1+3\alpha)] \in I,$

$$|y(t)| \leq |f(t)| + \frac{|\lambda|\bar{M}_1^{\alpha}}{\Gamma(\alpha)} t^{\alpha(2\alpha+1)-1}\left[\bar{\mathcal{A}}_1^{*1-\alpha}(t) + \frac{|\lambda|\bar{M}_1^{\alpha}}{(1-\alpha)\Gamma(\alpha)} \bar{\mathrm{V}}_1^{*-1}(\mathrm{t})\right.$$

$$\left.\times \left(\int_0^t s^{\frac{\alpha(2\alpha+1)-1}{1-\alpha}}(s)\, \bar{\mathcal{A}}_1^{*}(s) \bar{V}_1^{*}(s)ds\right)^{1-\alpha}\right] 0 > t\,, \qquad (27)$$

where

$$\bar{M}_1 = \frac{1}{2\alpha} B\left[\frac{3\alpha(2\alpha + 1) - 1}{2\alpha^2}, \frac{2\alpha^2 - 1}{\alpha}\right], \quad \bar{\mathcal{A}}_1^*(t) = \int_0^t |f(s)|^{\frac{1}{1-\alpha}} ds$$

and

$$\bar{V}_1^*(t) = exp\left[-\frac{1-\alpha}{2\alpha^2}\left(\frac{\bar{M}_1^{\alpha}|\lambda|}{(\Gamma(\alpha))}\right)^{\frac{1}{1-\alpha}} t^{\frac{2\alpha^2}{1-\alpha}}\right],$$

(ii) if $[\alpha, 2\alpha, 2\alpha(1 + 3\alpha)] \in II,$

$$|y(t)| \leq |f(t)| + \frac{|\lambda|\bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}}}{\Gamma(\alpha)} t^{\alpha(13\alpha+4)} \left[\bar{\mathcal{A}}_2^{*\frac{\alpha}{1+4\alpha}}(t) + \frac{\bar{M}_2^{\frac{1+3\alpha}{1+4\alpha}}(1+4\alpha)|\lambda|}{\alpha\Gamma(\alpha)}\right.$$

$$\left. \times\, \bar{\mathrm{V}}_2^{*-1}(\mathrm{t}) \left(\int_0^t s^{2\alpha(4\alpha+1)-1}(s)\, \bar{\mathcal{A}}_2^*(s) \bar{V}_2^*(s) ds\right)^{\frac{\alpha}{1+4\alpha}}\right], \quad t > 0, \quad (28)$$

where

$$\bar{M}_2 = \frac{1}{2\alpha} B\left[\frac{2\alpha(12\alpha + 7) + 1}{2(1 + 4\alpha)}, \frac{4\alpha^2}{1 + 3\alpha^2}\right],$$

$$\bar{\mathcal{A}}_2^*(t) \int_0^t |f(s)|^{\frac{1+4\alpha}{\alpha}}(s) ds$$

and

$$\bar{V}_2^*(t) = exp\left[-\frac{\bar{M}_2^{\frac{1+3\alpha}{\alpha}}}{2\alpha(1 + 4\alpha)}\left(\frac{|\lambda|}{\Gamma(\alpha)}\right)^{\frac{1+4\alpha}{\alpha}} t^{2\alpha(1+4\alpha)}\right].$$

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