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Lie Group invariant-solution for minimal surface equation

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Abstract:

One of the typical applications of Lie symmetry methods in the study of differential equations is the searching for symmetry-invariant solutions. We will present a review of some of the principal techniques related to this idea to obtain invariant solution of nonlinear second order PDE, and we will focus on minimal surface equation (MSE).

Keywords: group invariant solution, Lie group method of transformations, minimal surface equation, optimal system.

1 Introduction:

One of the most important discoveries of Sophus Lie, in differential equations is to show that, it is possible to transform non-linear conditions in a system, to linear conditions, by infinitesimal invariants, corresponding to the symmetry group generators, of the system [1].

The most powerful tools for studying differential equations (DEs), either ordinary or partial, is provided by the theory of symmetries (see Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 , 15,

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16]). Symmetries of DEs are (finite or infinitesimal) transformations of the independent and dependent variables and derivatives of the latter with respect to the former, with the further property of sending solutions into solutions.

In fact, the original purpose of what we now call Lie group symmetry was to use continuous groups to solve differential equations (ordinary and partial).

The history of minimal surfaces begins with J. L. Lagrange. Lagrange developed his algorithm for the calculus of variations, an algorithm which is also applicable in higher dimensions and which leads to what is known today as the Euler-Lagrange differential equation.

2 Lie point symmetry of MSE:

Minimal surfaces are defined as surfaces with zero mean curvature. Consider the following nonlinear second order PDE as MSE

$$\left(1+u_y^2\right)u_{xx}-2u_xu_yu_{xy}+\left(1+u_x^2\right)u_{yy}=0 \qquad (1)$$

which will be written in the form:

$$F\left(x,t,u^{(2)}\right)=\left(1+u_y^2\right)u_{xx}-2u_xu_yu_{xy}+\left(1+u_x^2\right)u_{yy} \qquad (2)$$

Equation (2) has Jacobian matrix with maximal rank one of the form:

$$J_F=\left(0,0,0,2u_xu_{yy}-2u_yu_{xy},2u_yu_{xx}-2u_xu_{xy},1+u_y^2,-2u_xu_y,1+u_x^2\right). \qquad (3)$$

Now we will have the following symmetry condition

$$\Pi^{\mathrm{x}}\left(2u_xu_{yy}-2u_yu_{xy}\right)+\Pi^{\mathrm{y}}\left(2u_yu_{xx}-2u_xu_{xy}\right)+\Pi^{\mathrm{xx}}\left(1+u_y^2\right)+\Pi^{\mathrm{xy}}\left(-2u_xu_y\right)+\Pi^{\mathrm{yy}}\left(1+u_x^2\right)=0 \qquad (4)$$

where MSE (1) hold

After substituting the expressions Π^x ,Π^y ,Π^{xx} , Π^{xy} and Π^{yy} into (4), this becomes

$$\left(\eta_{\mathrm{x}}+(\eta_{\mathrm{u}}-\xi_x)u_x-\varphi_xu_y-\varphi_uu_x\,u_y-\xi_uu_x^2\right)\left(2u_xu_{yy}-2u_yu_{xy}\right)$$
$$+$$
$$\left(\eta_{\mathrm{y}}+\left(\eta_{\mathrm{u}}-\varphi_y\right)u_y-u_x\,\xi_y-\xi_uu_y\,u_x-\varphi_uu_y^2\right)\left(2u_yu_{xx}-2u_xu_{xy}\right)$$
$$+$$
$$\left(\eta_{xx}+(2\eta_{xu}-\xi_{xx})u_x+(\eta_u-2\xi_x)u_{xx}+(\eta_{uu}-2\xi_{xu})u_x^2-\varphi_{xx}u_y-2\varphi_{xu}u_y\,u_x-\varphi_uu_y\,u_{xx}-\varphi_{uu}u_y\,u_x^2-\xi_{uu}u_x^3-2\varphi_xu_{yx}-\right.$$

$$\begin{gathered}2\varphi_u u_x u_{yx} - 3\xi_u u_x u_{xx})(1 + u_y^2) \qquad +(\eta_{xy} + (\eta_{\mathrm{uy}} - \xi_{xy})u_x + \\ (\eta_{ux} - \varphi_{xy})u_y - \xi_{uy}u_x^2 + (\eta_{uu} - \xi_{ux} - \varphi_{\mathrm{uy}})u_x u_y - \varphi_{ux}u_y^2 - \\ \xi_y u_{xx} + (\eta_u - \varphi_y - \xi_{\mathrm{x}})u_{xy} - \varphi_x u_{yy} - \xi_u u_y u_{xx} - 2\varphi_u u_y u_{xy} - \\ 2\xi_u u_x u_{xy} - \varphi_u u_x u_{yy} - \xi_{uu}u_x^2 u_y - \varphi_{uu}u_x u_y^2)(-2u_x u_y) + \\ (\eta_{yy} + (2\eta_{uy} - \varphi_{yy})u_y + (\eta_u - 2\varphi_y)u_{yy} + (\eta_{uu} - 2\varphi_{uy})u_y^2 - \\ \xi_{yy}u_x - 2\xi_{uy}u_x u_y - \xi_u u_x u_{yy} - \xi_{uu}u_x u_y^2 - \varphi_{uu}u_y^3 - 2\xi_y u_{xy} - \\ 2\xi_u u_y u_{xy} - 3\varphi_u u_y u_{yy})(1 + u_x^2) = 0 \qquad (5)\end{gathered}$$

by rearrange (5), and using MSE (1) when it is necessary we get the following polynomial equation in the variables $u_x, u_y, u_x^2, u_y^2, \ldots,$

$$\begin{gathered}\eta_{xx} + \eta_{yy} + (2\eta_{xu} - \xi_{xx} - \xi_{yy})u_x + (2\eta_{uy} - \varphi_{yy} - \varphi_{xx})u_y \\ +(\eta_{uu} + \eta_{yy} - 2\xi_{xu})u_x^2 + (\eta_{xx} + \eta_{uu} - 2\varphi_{uy})u_y^2 \\ -2(\eta_{xy} + \varphi_{xu} + \xi_{uy})u_y u_x - (\xi_{yy} + \xi_{uu})u_x^3 - (\varphi_{xx} + \varphi_{uu})u_y^3 \\ +(2\varphi_{xy} - \xi_{xx} - \xi_{uu})u_x u_y^2 + (2\xi_{xy} - \varphi_{yy} - \varphi_{uu})u_y u_x^2 \\ + 2(\varphi_y - \eta_u)u_{xx} \\ +2(\xi_x - \eta_u)u_{yy} - 2(\varphi_x + \xi_y)u_{xy} + 2(\eta_{\mathrm{y}} + \varphi_u)u_y u_{xx} \\ +2(\eta_{\mathrm{x}} + \xi_u)u_x u_{yy} - 2(\eta_{\mathrm{y}} + \varphi_u)u_x u_{yx} - 2(\eta_{\mathrm{x}} + \xi_u)u_y u_{xy} = 0 \\ (6)\end{gathered}$$

which all coefficients should be equal to zero. Therefore, we have the following linear system of partial differential equations for functions ξ, η and φ of the infinitesimal operator, called the determining equations for the symmetry group of the given equation:

Table 1: determining equations of MSE

$2\eta_{xu} - \xi_{xx} - \xi_{yy} = 0$	$\eta_{xx} + \eta_{yy} = 0$	$2\eta_{uy} - \varphi_{yy} - \varphi_{xx} = 0$
$\eta_{uu} + \eta_{yy} - 2\xi_{xu} = 0$	$\varphi_{xx} + \varphi_{uu} = 0$	$\eta_{xy} + \varphi_{xu} + \xi_{uy} = 0$
$2\varphi_{xy} - \xi_{xx} - \xi_{uu} = 0$	$\xi_x - \eta_u = 0$	$\xi_{yy} + \xi_{uu} = 0$
$2\xi_{xy} - \varphi_{yy} - \varphi_{uu} = 0$	$\varphi_y - \eta_u = 0$	$\varphi_x + \xi_y = 0$
$\eta_{xx} + \eta_{uu} - 2\varphi_{uy} = 0$	$\eta_{\mathrm{y}} + \varphi_u = 0$	$\eta_{\mathrm{x}} + \xi_u = 0$

solving the system of PDEs in Table 1, we will have the following general solution

$$\xi(x, y, u) = c_4 x - c_1 y + c_5 u + c_6, \qquad (7)$$

$$\varphi(x, y, u) = c_1 x + c_4 y + c_2 u + c_3, \qquad (8)$$

$$\eta(x, y, u) = -c_5 x - c_2 y + c_4 u + c_7, \qquad (9)$$

where $c_i \in \mathbb{R}$,$i = 1, \dots, 7$ are arbitrary constants. Then the Lie algebra of infinitesimal generators of MSE (1) is spanned by the seven vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial u}, \quad V_4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$V_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \; V_6 = u\frac{\partial}{\partial y} - y\frac{\partial}{\partial u}, \quad V_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} \quad (10)$$

These vector fields represent a basis for the 7-parameter symmetry vector field. And they produce a Lie algebra space $\mathscr{L}$ with the following commutator(Lie bracket) Table:

Table 2: the commutator table

$[V_i, V_j]$	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	0	0	V_2	$-V_3$	0	V_1
V_2	0	0	0	$-V_1$	0	$-V_3$	V_2
V_3	0	0	0	0	V_1	V_2	V_3
V_4	$-V_2$	V_1	0	0	$-V_6$	V_5	0
V_5	V_3	0	$-V_1$	V_6	0	$-V_4$	0
V_6	0	V_3	$-V_2$	$-V_5$	V_4	0	0
V_7	$-V_1$	$-V_2$	$-V_3$	0	0	0	0

3 Transformation for group invariant solutions of MSE

To obtain the group transformation which is generated by the infinitesimal generators V_i for $i = 1, \dots, 7$ we need to solve the three systems of first order ordinary differential equations

$$\begin{cases} \frac{\partial \bar{x}(s)}{\partial s} = \xi_i(\bar{x}(s), \bar{y}(s), \bar{u}(s)), \; \bar{x}(0) = x \\ \frac{\partial \bar{y}(s)}{\partial s} = \varphi_i(\bar{x}(s), \bar{y}(s), \bar{u}(s)), \; \bar{y}(0) = y \; i = 1, \dots, 7 \\ \frac{\partial \bar{u}(s)}{\partial s} = \eta_i(\bar{x}(s), \bar{y}(s), \bar{u}(s)), \; \bar{u}(0) = u \end{cases} \qquad (11)$$

By solving this system of ODEs, the one parameter group of $G_i(s): M \to M$ generated by V_i for $i = 1, \dots, 7$ is obtained.

For instance consider $V_1 = \frac{\partial}{\partial x}$ its associated system is given by

$$\bar{x}'(s) = 1, \qquad \bar{y}'(s) = 0, \qquad \bar{u}'(s) = 0 \qquad (12)$$

Now by integrating this system we get

$$\bar{x}(s) = s + A, \qquad \bar{y}(s) = B, \qquad \bar{u}(s) = C \qquad (13)$$

where A, B, C are constants. Using initial condition $(\bar{x}, \bar{t}, \bar{u})|_{s=0} = (x, t, u)$ we find $A = x$, $B = y$, $C = u$ then

$$(\bar{x}, \bar{y}, \bar{u}) = (s + x, y, u) \tag{14}$$

Take another example, for instance $V_4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ where

$$\bar{x}'(s) = -\bar{y}, \qquad \bar{y}'(s) = \bar{x}, \qquad \bar{u}'(s) = 0 \tag{15}$$

differentiating $\bar{x}'(s) = -\bar{y}$ with respect to s and substitute $\bar{y}'(s) = \bar{x}$ on it, we get

$$\bar{x}''(s) + \bar{x}(s) = 0 \tag{16}$$

then integrating it, we find

$$\bar{x} = A\cos(s) + B\sin(s) \tag{17}$$

where A, B are constants, using initial data condition $\bar{x}(0) = x$, one can get $A = x$, and to get B, differentiate (17) with respect to s and substitute $\bar{x}'(s) = -\bar{y}$ we find $B = -y$, then substituting A, B into (17) we obtain

$$\bar{x} = x\cos(s) - y\sin(s), \tag{18}$$

similarly

$$\bar{y} = y\cos(s) + x\sin(s), \qquad \bar{u} = u \tag{19}$$

thus form (18) and (19) we have the transformation

$$(\bar{x}, \bar{y}, \bar{u}) = (x\cos(s) - y\sin(s), y\cos(s) + x\sin(s), u) \tag{20}$$

which is a transformation of the original variables that preserves the equation, and so on.

All the calculations are summarized as following.

$$\begin{gathered}
G_1: (x, y, u) \mapsto (x + s, y, u) \\
G_2: (x, y, u) \mapsto (x, y + s, u) \\
G_3: (x, y, u) \mapsto (x, y, u + s) \\
G_4: (x, y, u) \mapsto (x\cos s - y\sin s, y\cos s + x\sin s, u) \\
G_5: (x, y, u) \mapsto (u\sin s - x\cos s, y, u\cos s + y\sin s) \\
G_6: (x, y, u) \mapsto (x, y\cos s - u\sin s, u\cos s + y\sin s) \\
G_7: (x, y, u) \mapsto (xe^s, ye^s, ue^s)
\end{gathered} \tag{21}$$

now, if $u = f(x, y)$ is one solution of MSE (1), then the following functions that have been produced through acting $G_i(s)$ on $u = f(x, y)$ will also be the solution of MSE (1).

$$\begin{gathered}
G_1(s) \cdot f(x, y) = f(x - s, y) \\
G_2(s) \cdot f(x, y) = f(x, y - s) \\
G_3(s) \cdot f(x, y) = f(x, y) + s \\
G_4(s) \cdot f(x, y) = f(x\cos s - y\sin s, y\cos s + x\sin s)
\end{gathered} \tag{22}$$

$$\left(G_5(s)\cdot f(x,y)\right) cos\ s + y\sin s = f\left(\left(G_5(s)\cdot f(x,y)\right) sin\ s - x\ cos\ s, y\right)$$

$$\left(G_6(s)\cdot f(x,y)\right) cos\ s + y\sin s = f\left(x, y\ cos\ s - \left(G_6(s)\cdot f(x,y)\right)\sin s\right)$$

$$G_7(s)\cdot f(x,y) = e^s f(xe^{-s}, ye^{-s}) \qquad (23)$$

4 Optimal system of one-dimensional subalgebras of MSE

Given a nonzero vector

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6 + a_7V_7 \qquad (23)$$

which depends on the seven arbitrary constants $a_1, \dots, a_1 \in \mathbb{R}$; to construct an optimal system of one-dimensional subalgebras of the Lie algebra $\mathfrak{g}$ we follow the method used by Olver [14] in adjoint representation

$$Ad(\mathrm{e}^{sV_i})V_j = V_j - s[V_i, V_j] + \frac{s^2}{2!}\left[V_i, [V_i, V_j]\right] - \frac{s^3}{3!}\left[V_i, \left[V_i, [V_i, V_j]\right]\right] + \cdots \qquad (24)$$

where the commutator of V_i and V_j is defined in Table 2. For illustration we show calculations in the following examples:

$$Ad\left((\mathrm{e}^{sV_4})\right)V_1 = V_1 - s[V_4, V_1] + \frac{s^2}{2!}[V_4, [V_4, V_1]] - \frac{s^3}{3!}\left[V_4, [V_4, [V_4, V_1]]\right] + \cdots$$
$$= V_1 + sV_2 - \frac{s^2}{2!}V_1 - \frac{s^3}{3!}V_2 + \frac{s^4}{4!}V_1 + \frac{s^5}{5!}V_2 + \cdots$$
$$= V_1\left(1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \cdots\right) + V_2\left(s - \frac{s^3}{3!} + \frac{s^5}{5!} - \cdots\right)$$
$$= V_1 cos(s) + V_2 sin(s) \qquad (25)$$

$$Ad\left((\mathrm{e}^{sV_7})\right)V_2 = V_2 - s[V_7, V_2] + \frac{s^2}{2!}[V_7, [V_7, V_2]] - \frac{s^3}{3!}\left[V_7, [V_7, [V_7, V_2]]\right] + \cdots = V_2 + sV_2 + \frac{s^2}{2!}V_2 + \frac{s^3}{3!}V_2 + \frac{s^4}{4!}V_2 + \cdots = V_2\left(1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \cdots\right) = V_2e^s \qquad (26)$$

and so on.

All the calculations are summarized in Table 3.

Table 3: adjoint representation

Ad	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	V_3
V_3	V_1	V_2	V_3
V_4	$V_1 cos(s) + V_2 sin(s)$	$V_2 cos(s) - V_1 sin(s)$	V_3
V_5	$V_1 cos(s) - V_3 sin(s)$	V_2	$V_3 cos(s) + V_1 sin(s)$
V_6	V_1	$V_2 cos(s) - V_3 sin(s)$	$V_3 cos(s) + V_2 sin(s)$
V_7	$V_1 e^s$	$V_2 e^s$	$V_3 e^s$

Ad	V_4	V_5	V_6	V_7
V_1	$V_4 - sV_2$	$V_5 + sV_3$	V_6	$V_7 - sV_1$
V_2	$V_4 + sV_1$	V_5	$V_6 + sV_3$	$V_7 - sV_2$
V_3	V_4	$V_5 - sV_1$	$V_6 - sV_2$	$V_7 - sV_3$
V_4	V_4	$V_5 cos(s) + V_6 sin(s)$	$V_6 cos(s) - V_5 sin(s)$	V_7
V_5	$V_4 cos(s) - V_6 sin(s)$	V_5	$V_6 cos(s) + V_4 sin(s)$	V_7
V_6	$V_4 cos(s) + V_5 sin(s)$	$V_5 cos(s) - V_4 sin(s)$	V_6	V_7
V_7	V_4	V_5	V_6	V_7

It remains to use the adjoint table to simplify as much as possible the

constants in equation (23). If V is given as in (23), and assume $a_7 = 1$ then

$$\widetilde{V} = \sum_{i=1}^{7} \widetilde{a}_i V_i = Ad\big(\exp(\alpha V_4)\big) \circ Ad\big(\exp(\beta V_5)\big)V \qquad (27)$$

This means referring to Table 3, we firstly act on (23) by $Ad\big(\exp(\alpha V_4)\big)$ to obtain

$$\acute{V} = \big(a_1 cos(\alpha) - a_2 sin(\alpha)\big)V_1 + \big(a_1 sin(\alpha) + a_2 cos(\alpha)\big)V_2 + a_3 V_3 + a_4 V_4 + \big(a_5 cos(\alpha) - a_6 sin(\alpha)\big)V_5 + \big(a_5 sin(\alpha) + a_6 cos(\alpha)\big)V_6 + V_7 \qquad (28)$$

acting on (28) by $Ad\big(\exp(\beta V_5)\big)$ we obtain

$$\widetilde{V} = \Big(cos(\beta)\big(a_1 cos(\alpha) - a_2 sin(\alpha)\big) + a_3 sin(\beta)\Big)V_1 + \big(a_1 sin(\alpha) + a_2 cos(\alpha)\big)V_2 + \Big(-sin(\beta)\big(a_1 cos(\alpha) - a_2 sin(\alpha)\big) + a_3 cos(\beta)\Big)V_3 + \Big(a_4 cos(\beta) + sin(\beta)\big(a_5 sin(\alpha) + a_6 cos(\alpha)\big)\Big)V_4 + \big(a_5 cos(\alpha) - a_6 sin(\alpha)\big)V_5 + cos(\beta)(\widetilde{a}_5)V_6 + a_7 V_7 \qquad (29)$$

This has the coefficients

$$\begin{cases}\tilde{a}_1 = \left(\cos(\beta)\left(a_1\cos(\alpha) - a_2\sin(\alpha)\right) + a_3\sin(\beta)\right) \\ \tilde{a}_2 = \left(a_1\sin(\alpha) + a_2\cos(\alpha)\right) \\ \tilde{a}_6 = \cos(\beta)\left(a_5\sin(\alpha) + a_6\cos(\alpha)\right)\end{cases} \quad (30)$$

if we assume $\cos(\beta) = 0$ and $\cos(\alpha) = -\frac{a_1\sin(\alpha)}{a_2}$, This implies

$\tilde{a}_2 = \tilde{a}_6 = 0$ and $\tilde{a}_1 \neq 0$. We can now suppose $\tilde{a}_1 = 1$, thus, V is equivalent to the vector

$$\tilde{V} = V_1 + \tilde{a}_3V_3 + \tilde{a}_4V_4 + \tilde{a}_5V_5 + V_7 \quad (31)$$

acting on (31) by $Ad\left(\exp(sV_6)\right)$ we obtain

$$\tilde{V} = V_1 + \tilde{a}_3\cos(s)V_3 + (\tilde{a}_4\cos(s) - \tilde{a}_5\sin(s))V_4 + (\tilde{a}_4\sin(s) + \tilde{a}_5\cos(s))V_5 + V_7 \quad (32)$$

Let $\cos(s) = 0$ then V_3 in $\tilde{V}$ vanish. We get

$$\tilde{\tilde{V}} = V_1 + \tilde{\tilde{a}}_4V_4 + \tilde{\tilde{a}}_5V_5 + V_7 \quad (33)$$

Continuous acting on (33) by the group generated by $V_5$and V_4 we have

V_4 and V_5 in $\tilde{\tilde{V}}$ vanish. No further simplifications are possible, as such V is equivalent to $V_1 + V_7$. If we now suppose $a_1 = a_7 = 0$, we have

$$V = a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6 \quad (34)$$

We can now assume $a_6 \neq 0$ and suppose $a_6 = 1$. This gives

$$V = a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + V_6 \quad (35)$$

Acting on V by adjoint maps generated by V_2 and V_3. Namely $Ad\left(\exp(-a_3V_2)\right)$ and $Ad\left(\exp(a_2V_3)\right)$ as such V_3 and V_2 in

V vanish. So that

$$\tilde{V} = \tilde{a}_4V_4 + \tilde{a}_5V_5 + V_6 \quad (36)$$

By acting on $\tilde{V}$ by the group generated by V_4 we obtain

$$\tilde{V} = \tilde{a}_4V_4 + (\tilde{a}_5\cos(s) - \sin(s))V_5 + V_6 \quad (37)$$

Let $\tilde{a}_5\cos(s) = \sin(s)$ then V_5 vanish. We find that V is equivalent to a

multiple of $V_6 + aV_4$, $a \in \mathbb{R}$. If we now take $a_6 = 0$ in (34) we have

$$V = a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 \quad (38)$$

We can now assume $a_5 \neq 0$ and suppose $a_5 = 1$. This gives

$$V = a_2V_2 + a_3V_3 + a_4V_4 + V_5 \quad (39)$$

Acting on V by adjoint maps generated by V_1. i.e. $Ad\left(\exp(-a_3V_1)\right)$ as such V_3 in V vanish. So that

$\widetilde{V} = \widetilde{a}_2 V_2 + \widetilde{a}_4 V_4 + V_5$ (40)

Acting on $\widetilde{V}$ by $Ad\left(\exp\left(\frac{\widetilde{a}_2}{\widetilde{a}_4} V_1\right)\right)$, therefore V_2 vanish. Continuous acting on

$\widetilde{V}$ by $Ad(\exp(sV_6))$ we have

$\widetilde{V} = (\widetilde{a}_4 \cos(s) - \sin(s))V_4 + V_5$ (41)

Let $\widetilde{a}_4 \cos(s) = \sin(s)$ then V_4 vanish. This means V is equivalent to V_5.

If we now take $a_5 = 0$ in (37) we have

$V = a_2 V_2 + a_3 V_3 + a_4 V_4$ (42)

We can now assume $a_4 \neq 0$ and suppose $a_4 = 1$. This gives

$V = a_2 V_2 + a_3 V_3 + V_4$ (43)

Acting on V by adjoint maps generated by V_1. i.e. $Ad(\exp(a_2 V_1))$ as such V_2 in V vanish, and acting on V by $Ad(\exp(sV_7))$ we obtain $\widetilde{V} = a_3 e^s V_3 + V_4$. Depending on the sign of a_3, we can make the coefficient of V_3 to be either $+1, 1 \ or \ 0$. Therefore V is equivalent to $V_4 + V_3, V_4 - V_3 \ or \ V_4$. If we now take $a_4 = 0$ in (42) we have

$V = a_2 V_2 + a_3 V_3$ (44)

We can now assume $a_3 \neq 0$ and suppose $a_3 = 1$. This gives

$V = a_2 V_2 + V_3$ (45)

Acting on V by adjoint maps generated by V_1. i.e. $Ad(\exp(a_2 V_1))$ as such V_2 in V vanish. Therefore V is equivalent to V_3.

The only remaining case is if $a_3 = 0$. This means V is equivalent to V_2.

The set of one-dimensional optimal system for (10) is given by following vector fields:

$V_1 + V_7, \ V_6 + aV_4, \ V_5, \ V_4 + V_3, \ V_4 - V_3, \ V_4, \ V_2, a \in \mathbb{R}$ (46)

Note: we can get more examples of optimal system by different calculations

5 Construction of invariant solutions for MSE

In this section we begin by discussing the classical notion of a group-invariant solution, which includes many of the common special solutions to partial differential equations, such as similarity solutions, travelling wave solutions, etc. we focus on similarity solution for MSE.

Now we will begin illustrating the construction of invariant solutions corresponding to the symmetry generators.

The symmetry variables are found by solving the invariant surface condition

$$\boldsymbol{\Phi} \equiv \mathbf{V} = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial y} - \eta = 0 \quad (47)$$

or the corresponding characterstic equations

$$\frac{dx}{\xi} = \frac{dy}{\varphi} = \frac{du}{\eta} \quad (48)$$

1. Invariance under The operator:

$$V_2 = \frac{\partial}{\partial y} \quad (49)$$

This operator has the characteristic system

$$\frac{dx}{0} = \frac{dy}{1} = \frac{du}{0} \quad (50)$$

There are two linear equations that can be formed from the above charac-

teristic system. The first equation is

$$\frac{dx}{0} = \frac{dy}{1} \quad (51)$$

Integrating the above equation yields $x = C_1$ where C_1 is a constant of inte-gration. Hence, one of the invariants is

$$\tau = x \quad (52)$$

If we check for the invariant τ we have

$$V_2(\tau) = \frac{\partial \gamma}{\partial y} = \frac{\partial x}{\partial y} = 0 \quad (53)$$

This operator satisfies the invariant condition.

Similarly, the integration of the equation

$$\frac{dy}{1} = \frac{du}{0} \quad (54)$$

yields $u = C_2$ where C_2 is the constant of integration. Therefore the second

invariant is

$$J = u \quad (55)$$

Similarly, check for the invariant J we have

$$V_2(J) = \frac{\partial J}{\partial y} = \frac{\partial u}{\partial y} = 0 \quad (56)$$

which also satisfied the invariant condition. Designating one of the invariants as a function of the other, that is

$$J = z(\tau) \quad (57)$$

we obtain

$$u = z(x) \quad (58)$$

Taking derivative of equation (58) with respect to x and y we obtain

$$u_x = z', \quad u_{xx} = z'', \quad u_{xy} = 0, \quad u_y = 0, \quad u_{yy} = 0 \qquad (59)$$

Setting the above derivatives into the MSE (1) we obtain

$$z'' = \frac{d^2 z}{d\tau^2} = 0 \qquad (60)$$

which is a second order ordinary differential equation, where a prime denotes differentiation with respect to the similarity variable τ.

with a solution given by

$$z(\tau) = a\tau + b, \qquad a, b \text{ are constant} \qquad (61)$$

and consequently the group invariant solution is given by

$$u(x, y) = ax + b \qquad (62)$$

Invariance under the operator:

$$V_1 + V_7 = (x + 1)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} \qquad (63)$$

This operator has the characteristic system

$$\frac{dx}{x+1} = \frac{dy}{y} = \frac{du}{u} \qquad (64)$$

There are two linear equations that can be formed from the above characteristic system. The first equation is

$$\frac{dx}{x+1} = \frac{dy}{y} \qquad (65)$$

Integrating the above equation yields $\frac{x+1}{y} = C_1$ where C_1 is constant of inte-gration. Hence, one of the invariants is

$$\tau = \frac{x+1}{y} \qquad (66)$$

If we check for the invariant J_1 we have

$$(V_1 + V_7)(\tau) = (x + 1)\frac{\partial \gamma}{\partial x} + y\frac{\partial \gamma}{\partial y} + u\frac{\partial \gamma}{\partial u} = (x + 1)\frac{\partial\left(\frac{x+1}{y}\right)}{\partial x}$$

$$+y\frac{\partial\left(\frac{x+1}{y}\right)}{\partial y} + u\frac{\partial\left(\frac{x+1}{y}\right)}{\partial u} = (x + 1)\frac{1}{y} - (x + 1)\frac{1}{y} + 0 = 0 \qquad (67)$$

This operator satisfies the invariant condition.Similarly, the integration of the equation

$$\frac{dy}{y} = \frac{du}{u} \qquad (68)$$

yields $u = C_2 y$ where C_2 is the constant of integration. Therefore the second invariant is

$$J = \frac{u}{y} \qquad (69)$$

Similarly, check for the invariant J we have

$$(V_1 + V_7)(J) = (x+1)\frac{\partial J_2}{\partial x} + y\frac{\partial J_2}{\partial y} + u\frac{\partial J_2}{\partial u} = (x+1)\frac{\partial\left(\frac{u}{y}\right)}{\partial x} + y\frac{\partial\left(\frac{u}{y}\right)}{\partial y} + u\frac{\partial\left(\frac{u}{y}\right)}{\partial u} = 0 - \frac{u}{y} + \frac{u}{y} = 0 \qquad (70)$$

which also satisfied the invariant condition. Designating one of the invariants as a function of the other, that is

$$J = z(\tau) \qquad (71)$$

we obtain

$$\frac{u}{y} = z(\tau\gamma) \Rightarrow u = yz(\tau) \qquad (72)$$

Taking derivative of equation (71) with respect to x and y we obtain

$$u_x = z', \;\; u_{xx} = \frac{z''}{y}, \;\; u_{xy} = -\frac{\tau z''}{y}, \;\; u_y = -\tau z' + z, \;\; u_{yy} = \frac{\tau^2 z''}{y} \qquad (73)$$

Setting the above derivatives into the MSE (1) we obtain

$$\frac{1}{y}\left(1 + (-\tau z' + z)^2\right)z'' + \frac{1}{y}2\tau z'(-\tau z' + z)z'' + \frac{1}{y}\tau^2\left(1 + {z'}^2\right)z'' = 0$$

$$\frac{1}{y}\left[\left(1 + (-\tau z' + z)^2\right) + 2\tau z'(-\tau z' + z) + \tau^2\left(1 + {z'}^2\right)\right]z'' = 0$$

$$[1 + z^2 + \tau^2]z'' = 0 \Rightarrow z'' = 0 \Rightarrow (\tau) = a\tau + b\,,\, a, b \in \mathbb{R} \qquad (74)$$

and consequently the group invariant solution is given by

$$\frac{u}{y} = a\left(\frac{x+1}{y}\right) + b \Rightarrow u(x,y) = a(x+1) + by \qquad (75)$$

and

$$1 + z^2 + \tau^2 = 0 \Rightarrow 1 + \left(\frac{u}{y}\right)^2 + \left(\frac{x+1}{y}\right)^2 = 0 \Rightarrow$$

$$y^2 + u^2 + (x+1)^2 = 0 \qquad (76)$$

also solution which satisfies MSE (1).

By this form we can get all invariant solutions of optimal system in (46).

6 Conclusions:

In this paper, we have studied MSE by investigating a nonlinear PDE using the Lie symmetry group method. We derived the Lie point symmetry generators of the nonlinear MSE. By classifying the Lie point symmetry generators, then we obtained the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the MSE. Next we used them to reduce the underlying MSE to a system of ODEs. group-invariant solutions of MSE are constructed from the reduced ODEs.

As the complexity of the equations under investigation increased, so recently the most amounts of the calculations involved are made by a computer algebra system employed for that task.

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