SJBS (M), (2016b), 34: 33-44 ISSN: 7615-1858

Various Properties of Hilbert Transform

Adam A. AbakarHassan (1) and Rugia M. Ahmed Adam (2)

Abstract:

In this paper we show basic properties of Hilbert transform that follows directly from the various definitions. We also show that any property comes directly from the fact that the Hilbert transform is the output of it's system.

الخلاصة:

فى هذه الورقة تم إيضاح الخواص الأساسية لتحويل هلبرتوالتى مباشرة تنبع من التعاريف المتنوعة .أيضاً أوضحنا ان أى خاصية تأتى مباشرة من الحقيقة ان تحويل هلبرتهو مخرج مننظامه.

Keywords: Cauchy principal- value, Hilbert Transform, Derivative, symmetry properties.

Introduction:

Clearly the Hilbert transform of a time-domain g*(t)* is another time-domain signal $g\hat{}$(t). If $g(t)$ real-valued then so is $\hat{}g(t)$[1].In this paper we look at some properties of the Hilbert transform. We assume that $F(\omega)$ does not contain any impulses for $f(\omega)$ = 0 and that $f(t)$ is the real valued function. Some of formulas are to be interpreted in a distributional sense.

1. Linearity

The Hilbert transform that is a Cauchy principle-valued function, is expressed on the form [2]

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(1) Nile Valley University,College of Edudation Department of mathematics, Sudan.

(2) Kordofan University, Virtual library, Sudan.

$Hf(t) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{f(t\ \ -\varepsilon_1)}{\varepsilon_1}d\tau.$

If we write the function $f(t)$ as $c_n f_n(t) + c_{n+1}f_{n+1}(t)$ where the Hilbert transform of $f_n(t)$ and $f_{n+1}(t)$ exists then for $\varepsilon_1 > 0$

$$Hf(t) = H(t(c_n f_n(t)) + c_{n+1}f_{n+1}(t))$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{(c_n f_n(t-\varepsilon_1)) + c_{n+1}f_{n+1}(t-\varepsilon_1)}{\varepsilon_1}d(t-\varepsilon_1)$$

$$= c_1\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{f_n(t)}{\varepsilon_1}d\,(t-\varepsilon_1) + c_{n+1}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{f_{n+1}(t)}{\varepsilon_1}d(t-\varepsilon_1)$$

$$= c_n Hf_n(t) + c_{n+1}Hf_{n+1}(t)$$

This is the linearity property of the Hilbert transform.

2. Multiple Hilbert transform and their inverses

The Hilbert transform used twice on a real function but with altered sign

$$H^2 = -I,$$

with$I$ as identity operator. The Hilbert transform used original function back

$$H^2H^2 = H^4 = 1. \qquad (2.1)$$

A more interesting property of multiple Hilbert transforms arises if we use the Hilbert transform 3 times

Thus

$$H^3H = 1 \Rightarrow H^{-1} = H^3 \qquad (2.2)$$

This tells us that it is possible to use multiple Hilbert transform to calculate the inverse Hilbert transform.

As we seen before Hilbert transform can be applied in the time domain by using the definition of Hilbert transform.[1,2]

By multiplying the Hilbert transform operator by itself we get an easy method to do multiple Hilbert transform, that is

$$H^n f(t) \overset{F}{\Leftrightarrow} (-i\ sgn(\omega)^n)\ F(\omega),$$

where n is the number of Hilbert transform.[1,2]

Example 2.1.[3] We want to calculate the inverse Hilbert transform of the function $f(t)$ by using multiple Hilbert transform in the frequency domain . First we take to Fourier transform of the function $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt,$$

And then use the Hilbert transform three times in the frequency domain, that is

$H^3 = (-i\, sgn(\omega)^3)$.

Finally we use the inverse Fourier transform

$$H^{-1}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^3 F(\omega)e^{i\omega t}(\omega)d\omega.$$

From above we see that we only have to calculate two infinite integrals in the frequency domain compared to three infinite integrals in the time domain.

Another advantage in the frequency domain is that we formally can choose the number of timesthat we want to use in the Hilbert transform.

3.Drivatives of the Hilbert transform

Theorem 3.1 : [3] The Hilbert transform of the derivative of a function is equivalent to the derivative of the Hilbert transform of function, that is

$$f(t) \overset{H}{\Leftrightarrow} \frac{d}{dt} f\hat{}(t) \qquad (3.1)$$

proof. From Definition (1) we have that $f\hat{}(t)$

$$\hat{f}(t) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{f(t-\varepsilon_1)}{\varepsilon_1} d(t-\varepsilon_1)$$

for$\varepsilon_1 > 0$

$$\hat{f}(t) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{f(t-\varepsilon_1)}{\varepsilon_1} d(t-\varepsilon_1)$$

and then if we apply the derivative of (t) on both sides we get

$$\frac{d}{dt} f\hat{}(t) = \frac{1}{\pi} p \int_{-\infty}^{\infty} \frac{f'(t-\varepsilon_1)}{\varepsilon_1} d(t-\varepsilon_1)$$

And the relation in (3.1) is valid .

From the proof above we conclude that the relation can be used repeatedly. Let us look at an example where we also use of multiple Hilbert transforms,

Example 3.2.

By (3.1) we may calculate the Hilbert transform of the delta function $\delta(t)$ and its derivatives. At the same time we get the Hilbert transform representation of the delta function. Consider the Hilbert transform of the delta function

$$2H\delta(t) = \frac{1}{\pi t}$$

The derivative of the delta function is calculated to be

$H\delta'(t) = -\frac{1}{\pi t^2}$ (3.2)

And if we apply the Hilbert transform on both sides then we get

$$\delta'(t) = H\left(\frac{1}{\pi t^2}\right)$$

The derivative of (3.1) is

$$H\delta''(t) = \frac{2}{\pi t^3}$$

And when we apply the Hilbert transform on both sides we get

$$H\delta''(t) = H\left(-\frac{2}{\pi t^2}\right)$$

.

This procedure can be continued.

4. Orthogonally properties

A symmetry about the Fourier transform $F(\omega)$ of a real function $f(t)$ leads us to the following definition[4,6]

Definition 4.1: A complex function is called Hermitian if its real part is even and its imaginary part is odd.

From this we have that the Fourier transform F(ω)of a real function $f(t)$is Hermitian.

Theorem 4.2: If $f(t)$is a real function that can be represented by inverse Fourier transform then we have the following relationship in the time domain

$$f(t) = f*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d(\omega)$$

$$= \int_{-\infty}^{\infty} F^*(\omega)e^{-i\omega t}d(\omega)$$

$$= \int_{-\infty}^{\infty} F^*(-\omega)e^{-i\omega t}d(\omega)$$

Theorem 4.3 : A real function $f(t)$ and its Hilbert transform $f^{\hat{}}(t)$ are orthogonal if f ,$f^{\hat{}}$ and F belonged to $L^1(\Re)$ or if $f \; and \; f^{\hat{}}$ belong to $L^2(\mathbf{\Re})$.

Proof From the Theorem (4.2) we have that

$$\int_{-\infty}^{\infty} f(t)f^{\hat{}}(t)dt \;=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\big(-i\,sgn(\omega)F(\omega)\big)d\omega$$

$$=\frac{i}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)\,F(\omega)F^{*}(\omega)d(\omega)$$

$$=\;\frac{i}{2\pi}\int_{-\infty}^{\infty} sgn(\omega)|F(\omega)|^2\,d\omega,$$

Where sgn(ω) is an odd function and the fact that $F(\omega)$ is Hermitian gives us that $|F(\omega)|^2$ is an even function. We conclude that

$$\int_{-\infty}^{\infty} f(t)f^{\hat{}}(t)dt \;=0,$$

and a real function and its Hilbert transform are orthogonal.

5. Energy aspects of Hilbert transform

The energy of a function $f(t)$ is closely related to the energy of its Fourier transform $F(\omega)$.The definitionof the energy[5,6] of $f(t) \; and \; F(\omega)$ is

$$E_f \;=\; \int_{-\infty}^{\infty} |f^{\hat{}}(t)|^2\,dt \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d(\omega). \;(5.1)$$

Here it is natural to assume that $f \;\in\; L^2(\Re)$ which means that E_f is finite. The same theorem is used to define the energy of Hilbert transform of $f(t) \; and \; F(\omega)$ that is

$$E_{f^{\hat{}}=} \int_{-\infty}^{\infty} |f^{\hat{}}(t)|^2\,dt \;=\frac{1}{2\pi}\int_{-\infty}^{\infty} |-i\,sng(\omega)F(\omega)|^2 d\omega, \;(5.2)$$

Where $|-i\,sgn(\omega)|^2 \;=\; 1$ except for ω = 0. But since $F(\omega)$ does not contain any impulses at the origin we get $E_{f^{\hat{}}=}\,E_{f.}$

A consequence of (5.1) is that $f \in L^2(\Re)$ induces that $f^{\hat{}} \in L^2(\Re)$. The accuracy of the approximated Hilbert transform operator can be measured by comparing the energy in (5.1) and (5.2). However, a minor difference in energy always exists in real applications due unavoidable truncation errors.

6. The Hilbert transform of strong analytic signal

Now we have that the Hilbert transform of strong analytic signal $z(t)$ is

$$Hz(t) = H\big(f(t) + i\hat{f}(t)\big) = \hat{f}(t) - if(t) = -iz(t) \quad (6.1)$$

We can show the result of the Hilbert transform of general multipliedstrong analytic signals induced from (6.1)[7]

Theorem 6.1: The product of $H(z_n(t)z_{n+1}(t))$is identical with the product of $z_n(t)H(z_{n+1}(t))$ if $z_n(t) and\, z_{n+1}(t)$ are strong analytic signals.

Proof: Since $z_n(t)\ and\ z_{n+1}(t)$ are strong analytic signals then

$$H(z_n(t))z_{n+1}(t) = \big(\hat{f}_n(t) - if_n(t)\big)\big(f_{n+1}(t) + i\hat{f}_{n+1}(t)\big) \quad (6.2)$$

$$= -i\big(f_n(t) + i\hat{f}_n(t)\big)\big(f_{n+1}(t) + \hat{f}_{n+1}(t)\big)$$

$$= -iz_n(t)z_{n+1}(t)$$

$$= \big(f_n(t) + if_{\,n}(t)\big)\big(\hat{f}_{n+1}(t) - i\hat{f}_{n+1}(t)\big) (6.3)$$

$$= z_n(t)H(Z_{n+1}(t)), \quad (6.4)$$

Where we make use of (6.1) in (6.2) and (6.3).

Theorem 6.2: The product of $z_n(t)z_{n+1}(t)$ are identical with the product

$$iH\big(z_n(t)\big)z_{n+1}(t) = iz_n(t)H(z_{n+1}(t))\ if\ z_{n+1}(t)\ and\ z_n(t)$$

are strong analytic signals.

Proof: Since $z_n(t) and\, z_{n+1}(t)$are strong analytic signals then

$$z_n(t)z_{n+1}(t) = \big(f_n(t) + i\hat{f}_n(t)\big)\big(f_{n+1}(t) + i\hat{f}_{n+1}(t)\big)$$

$$= i\big(\hat{f}_n(t) - i\hat{f}_n(t)\big)\big(f_{n+1}(t) + i\hat{f}_{n+1}(t)\big)$$

$$= iH(z_n(t)z_{n+1}(t) = iz_n(t)H(z_{n+1}(t)),$$

And the theorem follows.

The Hilbert transform of the product of two strong analytic signals gives us the same result as in (6.4). To prove this we first need to show that the product of two strong analytic signals is strong analytic.

Theorem 6.3: [8,10] $\mathrm{H}\big(z_n(t)z_n(t)\big) = iz_n(t)z_{n+1}(t)$

if $z_n(t)\ and\ z_{n+1}(t)$ are strong analytic signals.

Proof: Since $z_n(t) and\, z_{n+1}(t)$ are strong analytic signals then

$$H\big(z_n(t)z_{n+1}(t)\big) = H\big(f_n(t) + i\hat{f}_n(t)\big)\big(f_{n+1}(t) + i\hat{f}_{n+1}(t)\big)$$

$$= H\big(f_n(t)f_{n+1}(t) - \hat{f}_n(t)\hat{f}_{n+1}(t)\big)$$

$$+ i(f_n(t)\hat{f}_{n+1}(t) + \hat{f}_n(t)f_{n+1}(t)$$

$$= f_n(t)\hat{f}_{n+1}(t) + \hat{f}_n(t)f_{n+1}(t)$$

$$\begin{aligned}-i(f_n(t)f_{n+1}(t) - i\hat{f}_n(t)\hat{f}_{n+1}(t) \\ = -i(f_n(t)f_{n+1}(t) + i\hat{f}_n(t)\hat{f}_{n+1}(t) \\ +i\hat{f}_n(t)f_{n+1}(t) - \hat{f}_n(t)\hat{f}_{n+1}(t)) \\ = -i(f_n(t) + i\hat{f}_{n+1}(t))(f_n(t) + i\hat{f}_{n+1}(t)) \\ = -iz_n(t)z_{n+1}(t),\end{aligned} \tag{6.5}$$

Where we make use of (6.3) in(6.5)

Consequently it is possible to apply the Hilbert transform on product of two strong analytic signals in several different ways, thus

$$\begin{aligned}H\big(z_n(t)z_{n+1}(t)\big) = H\,\big(z_n(t)\big)z_{n+1}(t) = iz_n(t)z_{n+1}(t)z_{n+1}(t)) \\ = -iz_n(t)z_{n+1}(t)\end{aligned}$$

It does not matter on which strong analytic signal we apply the Hilbert transform. We conclude that the Hilbert transform of the product of m strong analytic signals from the equation

$$Hz^m(t) = H\big(z(t)\big)z^{m-1}(t) = -iz(t)z^{m-1}(t) = -iz^m(t)$$

7. Analytic signals in the time domain

The Hilbert transform can be used to create an analytic signal from areal signal. Instead of studying the signal in frequency to look at a rotating vector with an instantaneous phase $\varphi(t)$ and an instantaneous amplitude $A(t)$ in time domain, that is[9]

$$z(t) = f(t) + i\hat{f}(t) = e^{i\varphi(t)}.$$

This notation is usually called the polar notation where

$$A(t) = \sqrt{f^2(t)\hat{f}^2(t)},$$

And

$$\varphi(t) = arctan\left(\frac{\hat{f}(t)}{f(t)}\right)$$

If we express the phase with Taylor series then

$$\varphi(t) = \varphi(t + \varepsilon_2) + (\varepsilon_2)\varphi'(t + \varepsilon_2) + R,$$

Where R is small if $\varepsilon_2 > 0$. The analytic signal becomes

$$z(t) = A(t)e^{i\varphi(t)} = A(t)e^{i\left(\varphi(t+\varepsilon_2)-(t+\varepsilon_2)\varphi'(t+\varepsilon_2)\right)}e^{it\varphi'(t+\varepsilon_2)}e^{iR}.$$

And we see that $\varphi'(t + \varepsilon_2)$ has the role of frequency if R is neglected. This makes it natural to introduce the notation of instantaneous frequency, that is

$$\omega(t) = \frac{d\varphi(t)}{dt}.$$

Example 10: [10] We have a real signal and its Hilbert transform

$$f(t) = \cos(\omega_0 t),$$
$$\hat{f}(t) = \sin(\omega_0 t),$$

Together they form an analytic signal where the instantaneous amplitude is

$$A(t) = \sqrt{cos^2(\omega_0 t) + sin^2(\omega_0 t)} = 1.$$

The instantaneous frequency is easy to calculate form the phase $\varphi(t) = \omega_0 t$, that is

$$\omega(t) = \omega_{0.}$$

We see that in this particular case the instantaneous frequency is the same as the real frequency.

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