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Some Properties of Sequences of Symmeteric Meixner -Pollaczek Polynomials

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Abstract:

The sequence of the symmetric Meixner-Pollaczek polynomials $P_n^{(\lambda_r)}$for $\lambda_r > 0$ and $\lambda_r \leq 0$ are considered. These are polynomials orthogonal and respectively not orthogonal on the real line. We define a non-standard inner product with sequence of polynomials $P_n^{\lambda_r}$, for $\lambda_r \leq 0$ shared by $\lambda_r \geq 0$, for which it becomes a sequence of orthonormal polynomials.

Keywords: Meixner -Pollaczek polynomial, Orthogonal polynomial, Orthonormal polynomial. Polynomial operator, Non-standard inner product,

Introduction:

The Meixner-Pollaczek Polynomials were first invented by Meixiner [1]. Some of the main properties of these polynomials are presented in Pollaczek[2]. Erdélyi et al [3],Chihara [4], Askey and Wilson [5], and in the report by Koekoek and Swarttouw [6] with applications in the works of Rahman [7], Atakishiyen and Suslov[8], Bender et al. [9],Koornwinder [10], Li and Wong [11] and the improvements of Tsehaye K. Araaya [12]. We mainly consider the work of Tsehaye K. Araaya [12] and derive some properties of the modified sequence of the symmetric Meixner-Pollaczek Polynomials,$\left\{P_n^{\lambda_r}\right\}_{n,r=0}^{\infty}$.

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We denote the monic forms of these polynomials $P_n^{(\lambda_r)}(x) = n!\,P_n^{(\lambda_r)}(x)$.

The monic forms of the symmetric Meixner-Pollaczek polynomials are completely described by the recurrence formula [6].

$P_{-1}^{(\lambda_r)}(x) = 0,\ P_0^{(\lambda_r)}(x) = 1 \ \ where \ \ \lambda_r > 0,\ for\, r = 1,2,3,\ldots$(1)

and

$xP_n^{(\lambda_r)}(x) = P_{n+1}^{(\lambda_r)}(x) + n(n-1+2\lambda_r)P_{n-1}^{(\lambda_r)}(x),\ n, r = 1,2,3,\ldots$

since$\lambda_r > 0$ in the recurrence relation (1), the coefficient $n(n-1+2\lambda_r)$ in (1) is strictly positive for each $n \geq 1$. Thus according to Favard's condition [13] there exists a positive real measure μ such that these polynomials are orthogonal with respect to the inner product

$$\sum_{r=1}^{\infty}\left(P_n^{(\lambda_r)}, P_m^{(\lambda_r)}\right) = \int_{\mathbb{R}}\sum_{r=1}^{\infty} P_n^{(\lambda_r)}\, P_m^{*(\lambda_r)}\, d\mu \qquad (2)$$

For these polynomials a modified measure is known and is given by the following weight functions.

$$\omega_{\sum_{r=1}^{\infty}\lambda_r}(x) = \frac{\left|\Gamma\left(\sum_{r=1}^{\infty}\lambda_r + \frac{ix}{2}\right)\right|}{2\pi},\ \lambda_r > 0 \qquad (3)$$

What happens if $\sum_{r=1}^{\infty}\lambda_r \leq 0$, from the mentioned Favard's condition[12], there is no positive real measure μ, such that (2) yields an orthogonal system.

Duran [14], and Marcellán and Álvarez-Nodarse [15] tried to generalize Favard'sTheorem to include a wider class of polynomials satisfying certain recurrence relations. Tsehaye K. Araaya [12] find a general inner product, such that the sequence of polynomials with the deficient recurrence condition becomes an orthogonal system with respect to this inner product.

Inner-products other than the standard one are often used, particularly when a non-standard inner-product is more natural. Orthogonal polynomials with respect to such inner products can also be considered. For example, Sobolev type orthogonal polynomial appear in the works of Milovanović [16], Marcellán and Álvarez-Nodarse [15]. In general, the Sobolev type inner-product is defined by

$$(f, g) = \sum_{k=0}^{m}\int_{\mathbb{R}} f^{(k)}(t)g^{(k)}(t)d\mu_k(t), \qquad (4)$$

Where $d\mu_k(t), k = 0,1,\dots,m$, are given positive measures on$\mathbb{R}$.

Lately the main contribution in [12] was to show that the limiting case of the symmetric Meixner-P ollaczek polynomials,$P_n^{(0)}(x) = \lim_{\lambda\to 0^+} P_n^{(\lambda)}(x)$, is an orthogonal polynomial system on the strip $S = \{z: -1 \le \Im(z) \le 1\}$. In another contribution [17], the whole class of polynomials$P_n^{(\lambda)}(x)$for$\lambda \in \mathbb{R}$, was discussed in [18],[19],[20]. Besides the close relationship between the polynomials $P_n^{(\lambda)}(x)$, for $\lambda_r < 0$ and the Meixner-Pollaczek polynomials was exploited, and it was shown that for each fixed $\lambda_r \in \mathbb{R}, P_n^{(\lambda_r)}(x)$ is sheffer relative to the system $P_n^{(0)}(x)$.

Motivated by [21] and by the Sobolev type orthogonal polynomials [15],[16] corresponding to the Sobolev type inner product (4), Tsehaye K. Araaya [12] consider$\mathbb{P} = \left\{\left\{P_n^{(\lambda_r)}(x)\right\}_{n,r=0}^{\infty} : \lambda_r \in \mathbb{R}\right\}$, and for every $\lambda_r \in \mathbb{R}$, define an inner product with respect to which the system $\left\{P_n^{(\lambda_r)}(x)\right\}_{n,r=0}^{\infty}$ becomes orthogonal.For$\lambda > 0$ these inner products coincide with the standard inner products for the Meixner-Pollaczek polynomials.

Inner product is defined, and orthogonality of the system of polynomials with respect to this inner product are proved.We examine some of the major properties of these systems of polynomials share in common with the Meixner-Pollaczek polynomials.

Notations:

$\mathbb{N}$ is the set of positive integers,$\mathbb{Z}^-$ is the set negative integers,$\mathbb{Z}_0^- \equiv \mathbb{Z}^- \cup \{0\}$, $G_{\lambda_r}(x, t_r)$denote the generating function of the sequence of polynomials $\left\{P_n^{(\lambda_r)}(x)\right\}_{n,r=0}$, $\bar{x}$ denote the complex conjugate of the variable$x$, f^* is the complex conjugate of the function$f$.

Definition1. For each $\lambda_r \in \mathbb{R}$ the sequence of symmetric polynomials $\left\{P_n^{(\lambda_r)}(x)\right\}_{n,r=0}$ are defined by the following recurrence relation [6]

$P_{-1}^{(\lambda_r)}(x) = 0,\ P_0^{(\lambda_r)}(x) = 1,\ r = 1,2,3,\ldots$

And

$$(n+1)P_{n+1}^{(\lambda_r)}(x) - xP_n^{(\lambda_r)}(x) + (n-1+2\lambda_r)P_{n-1}^{(\lambda_r)}(x) = 0,\ n,r = 1,2,3,\ldots \quad (5)$$

where the monic forms are as described in (5). This sequence of polynomials has a generating function [6]

$$G_{\lambda_r}(x,t_r) = \frac{e^{x\tan^{-1}t_r}}{(1+t_r^2)\lambda_r} = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} P_n^{(\lambda_r)}(x)t_r^n \quad (6)$$

The system has a hypergeometric representation, in particular if $2\lambda_r \in \mathbb{R}\backslash\mathbb{Z}_0^-$, it is given by

$$P_n^{(\lambda_r)}(x) = \frac{(2\lambda_r)_n}{n!} i^n\ {}_2F_1\left(\left.\begin{matrix} -n, \lambda_r + \frac{ix}{2} \\ 2\lambda_r \end{matrix}\right| 2\right). \quad (7)$$

Consider the operators R and J defined in [17], [21] respectively, by

$$Rf(x) = \frac{f(x+i)+f(x-i)}{2} \quad (8)$$

$$Jf(x) = \frac{f(x+i)-f(x-i)}{2i} \quad (9)$$

Such that

$$R^2 + J^2 = I \quad (10)$$

where I is the identity operator and

$$(R \pm iJ)f(x) = f(x \pm i) \quad (11)$$

The following modified Proposition was proved in [17] in different setting.

Proposition 2. Given any $\lambda_r \in \mathbb{R}$, the following relations hold true:

$$RP_n^{(\lambda_r)}(x) = P_n^{(\lambda_r+\frac{1}{2})}(x) \quad (12)$$

$$JP_n^{(\lambda_r)}(x) = P_{n-1}^{(\lambda_r+\frac{1}{2})}(x) \quad (13)$$

The action of the operator R as in [21] is described as follows:

$R(fg^*) = \frac{f(x+i)g^*(x+i)+f(x-i)g^*(x-i)}{2} = \frac{f(x+i)\overline{g(x+\iota)}+f(x-i)\overline{g(x-\iota)}}{2} =$
$f(x-i)Rg(x) + iJf(x)g(x-i)$. (14)

(i) For any positive integer n, $R^{n+1}f = R[R^n f]$. Where
$R^n f = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} f\big(x + i(n-2k)\big)$. (15)

If $r = n + 1$, then $R^r\lambda_r = R\,[R^{r-1}\lambda_r]$ (16)

$R^{r-1}\lambda_r = \frac{1}{2^{r-1}}\sum_{k=0}^{r-1}\binom{r-1}{k}\lambda_r$

For any real number $\lambda_r \leq 0$ we can define, $\mathbb{N}_{\lambda_r} = \left\{n: n \in \mathbb{N} \text{ and } \lambda_r + \frac{n}{2} > 0\right\}$, then the following statement in [21]is always true.

Lemma3. $\mathbb{N}_{\sum_{r=2}^{\infty}\lambda_r}$**.**has a least element, where = 2,3,4, ...

We denote the associated element in Lemma 3 by $m_{\sum_{r=2}^{\infty}\lambda_r}$, where $m_{\sum_{r=2}^{\infty}\lambda_r} = \min_{n\in\mathbb{N}}\left\{n\colon \sum_{r=2}^{\infty}\lambda_r + \frac{n}{2} > 0\right\}$.

Note that $\sum_{r=1}^{\infty}\left(\lambda_r + \frac{m_{\lambda_r}}{2}\right) \in \left(0, \frac{1}{2}\right]$. Now we define the associate series of inner products as follows:

$(f, g)_{\sum_{r=0}^{\infty}\lambda_r,0} = \int_{-\infty}^{\infty} fg^*(P_{\sum_{r=0}^{\infty}\lambda_r,0}(x)) = \int_{-\infty}^{\infty} R^{m_{\Sigma\lambda_r}}(fg^*) + \omega_{\sum\lambda_r} + \frac{m_{\Sigma\lambda_r}}{2}(x)dx$ (17)

where

$\omega_{\sum\lambda_r} + \frac{m_{\Sigma\lambda_r}}{2} = \frac{\left|\Gamma(\lambda_r + \frac{m_{\Sigma\lambda_r}}{2} + \frac{ix}{2}\right|^2}{2\pi}$ (18)

Are the weight function associated with the polynomials $P_n^{\lambda_r + \frac{m_{\Sigma\lambda_r}}{2}}$ in the symmetric Mexiner-Pollaczek polynomials if $P_{\sum\lambda_r,0}(x) = I$, then we can deduce that $\|f\|_{\lambda_r,0} = (\int|f|^2)^{\frac{1}{2}}$, and $R^{m_{\Sigma\lambda_r}}$ is equivalent to an identity operator while the weight functions $\sum_{r=1}^{\infty}\left(\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)\right) \equiv 1$ for any $r = 1,2, \ldots$.

Result:

The following modified Theorem shown by [21].

Theorem4. For each $\lambda_r \leq 0$, the associated inner product defined in (17) is well defined, and the corresponding polynomial system$\left\{P_n^{(\lambda_r)}(x)\right\}_{n,r=0}$, is an orthogonal polynomial system with respect to this inner product.

Proof. For suitable real function f, g, or $\lambda_r^* = g^*$ series of inner product in (17) is equivalent to

$(f, g)_{\sum_r \lambda_r,0} = \int_{-\infty}^{\infty}\sum_{r=0}^{\infty} R^{m_{\lambda_r}}(fg^*) + \omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx$
$= \int_{-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{1}{2^{m_{\lambda_r}}}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k} \times f\left(x + i\left(m_{\lambda_r} - 2k\right)\right) g^*\left(x - i\left(m_{\lambda_r} - 2k\right)\right)\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx = \sum_{r=0}^{\infty}\frac{1}{2^{m_{\lambda_r}}}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k} \times \int_{-\infty}^{\infty} f\left(x + i\left(m_{\lambda_r} - 2k\right)\right) g\left(x - i\left(m_{\lambda_r} - 2k\right)\right)\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx$ (19)

Interchanging the order of summation and integration is permissible because $m_{\sum\lambda_r}$ is finite. Thus, it is easy to see that the inner product is well defined. Now we use the sequence of generating functions of the corresponding polynomials. Suppose $G_{\lambda_r}(x,\cdot)$is the sequence of generating functions corresponding toλ_r, then (see [12]).

$$\left\|\sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty} G_{\lambda_r}(x, t_\alpha)\right\|_{\lambda_r,0} = \sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\|P_n^{(\lambda_r)}(x)t_\alpha^m\right\|_{\lambda_r,0} \quad (20)$$

Which can also be equivalently described by

$$\left(\sum_{r=0}^{\infty}\sum_{\alpha=0}^{\infty} G_{\lambda_r}(x, t_\alpha), \sum_{r=0}^{\infty}\sum_{\alpha=0}^{\infty} G_{\lambda_r}(x, S_\alpha)\right) = \sum_{r=0}^{\infty}\sum_{\alpha=0}^{\infty}\int_{-\infty}^{\infty} R^{m_{\lambda_r}}\left(G_{\lambda_r}(x, t_\alpha), G_{\lambda_r}^*(x, S_\alpha)\right)\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx \quad (21)$$

Now we give our following results.

Corollary 5. Show that $\lambda_r = \frac{1}{2}\left(\frac{2x\tan^{-1}t_\alpha}{\ln(1+t_\alpha^2)} + 1\right), r = 1,2,3\ldots$

Proof. From [21] we have

$$R^{m_{\lambda_r}}\left(G_{\lambda_r}, G_{\lambda_r}^*\right) = \frac{1}{(1+t_\alpha^2)^{\frac{1}{2}}}\frac{1}{(1+S_\alpha^2)^{\frac{1}{2}}}(1 + t_\alpha S_\alpha)$$

Thus

$$R^{m_{\lambda_r}}\left(G_{\lambda_r}, G_{\lambda_r}^*\right) = \frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}} e^{x(tan^{-1}t_\alpha + tan^{-1}S_\alpha)}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}} \quad (22)$$

If $t_\alpha = S_\alpha$ we can deduce respectively that

$$R^{m_{\lambda_r}}\left(G_{\lambda_r}(x, t_\alpha), G_{\lambda_r}^*(x, t_\alpha)\right) = \frac{1}{(1+t_\alpha^2)}$$

and

$$R^{m_{\lambda_r}}\left(G_{\lambda_r}, G_{\lambda_r}^*\right) = \frac{e^{2xtan^{-1}t_\alpha}}{(1+t_\alpha^2)^{\lambda_r}}$$

on dividing we have

$$e^{2x\tan^{-1}t_\alpha}\frac{(1+t_\alpha^2)}{(1+t_\alpha^2)^{2\lambda_r}} = 1$$

$$e^{2x\tan^{-1}t_\alpha} = (1+t_\alpha^2)^{2\lambda_r-1}$$

$$2x\tan^{-1}t_\alpha = (2\lambda_r-1)\ln(1+t_\alpha^2)$$

$$2\lambda_r - 1 = \frac{2x\tan^{-1}t_\alpha}{\ln(1+t_\alpha^2)}$$

Hence

$$\lambda_r = \frac{1}{2}\left(\frac{2x\tan^{-1}t_\alpha}{\ln(1+t_\alpha^2)}+1\right), r = 1,2,3\ldots$$

Lemma 6. For the minimum n corresponding to m_{λ_r} we have to show that

$$n > -\left(\frac{2x\tan^{-1}t_\alpha}{\ln(1+t_\alpha^2)}+1\right)$$

Proof. For

$$\lambda_r + \frac{n}{2} > 0$$

We can find , for $n \in N$, that

$$n > -\left(\frac{2x\tan^{-1}t_\alpha}{\ln(1+t_\alpha^2)}+1\right)$$

Now from (21) and (22), [12] with a modification shows that

$$\left(G_{\lambda_r}(x,t_\alpha), G_{\lambda_r}(x,S_\alpha)\right)_{\lambda_r,0} = \int_{-\infty}^{\infty}\frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}}e^{x(\tan^{-1}t_\alpha+\tan^{-1}S_\alpha)}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}}\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx =$$

$$\frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}}\int_{-\infty}^{\infty}e^{x(\tan^{-1}t_\alpha+\tan^{-1}S_\alpha)}\omega_{\lambda_r} + \frac{m_{\lambda_r}}{2}(x)dx$$

$$= \frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}}\frac{2^{1-2\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}\Gamma(2\lambda_r+m_{\lambda_r})}{\cos^{2\lambda_r+m_{\lambda_r}}(\tan^{-1}t_\alpha+\tan^{-1}S_\alpha)} =$$

$$\frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}}\frac{2^{1-(2\lambda_r+m_{\lambda_r})}\Gamma(2\lambda_r+m_{\lambda_r})}{\frac{1}{(1+t_\alpha^2)^{\frac{1}{2}}}\frac{1}{(1+S_\alpha^2)^{\frac{1}{2}}}(1-t_\alpha S_\alpha)^{2\lambda_r+m_{\lambda_r}}} =$$

$$\frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}}}{(1-t_\alpha S_\alpha)^{2\lambda_r+m_{\lambda_r}}}2^{1-(2\lambda_r+m_{\lambda_r})}\Gamma(2\lambda_r+m_{\lambda_r})$$

$$= 2^{1-(2\lambda_r+m_{\lambda_r})}\Gamma(2\lambda_r + m_{\lambda_r})\sum_{n=0}^{\infty}(2\lambda_r + m_{\lambda_r})_n \frac{(t_\alpha S_\alpha)^n}{n!}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k}(t_\alpha S_\alpha)^k = 2^{1-(2\lambda_r+m_{\lambda_r})}\sum_{n=0}^{\infty}\sum_{k=0}^{m_{\lambda_r}}\Gamma(2\lambda_r + m_{\lambda_r} + n)\binom{m_{\lambda_r}}{k}\frac{(t_\alpha S_\alpha)^{n+k}}{n!} = 2^{1-(2\lambda_r+m_{\lambda_r})}\sum_{n=0}^{\infty}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k}\Gamma(2\lambda_r + m_{\lambda_r} + n - k)\frac{(t_\alpha S_\alpha)^n}{(n-k)!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k}\frac{2^{1-(2\lambda_r+m_{\lambda_r})}\Gamma(2\lambda_r+m_{\lambda_r}+n-k)}{(n-k)!}\right)(t_\alpha S_\alpha)^n \qquad (23)$$

Finally, comparing the coefficients of the powers of t_α and S_α (21) and (23) we observe that

$$\left(P_n^{(\lambda_r)}, P_m^{(\lambda_r)}\right)_{\lambda_r,0} = \delta_{nm}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k}\frac{2^{1-(2\lambda_r+m_{\lambda_r})}\Gamma(2\lambda_r+m_{\lambda_r}+n-k)}{(n-k)!} \quad (24)$$

Where δ_{nm} is the Kronecker delta function.

Remark 7. The above result can also be seen from the expansion of the expression in the right-hand side of (22).

$$\frac{(1+t_\alpha S_\alpha)^{m_{\lambda_r}} e^{x(\tan^{-1} t_\alpha+\tan^{-1} S_\alpha)}}{(1+t_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}(1+S_\alpha^2)^{\lambda_r+\frac{m_{\lambda_r}}{2}}} = (1 + t_\alpha S_\alpha)\sum_{n,r=0}^{\infty} P_n^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x)\, t_\alpha^n \sum_{m,r=0}^{\infty} P_m^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x)\, S_\alpha^m = \sum_{k=0}^{m_{\lambda_r}}$$

$$\sum_{r=0}^{\infty} \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k} P_n^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x) P_m^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x) t_\alpha^{n+k} S_\alpha^{m+k} = \sum_{n,m,r=0}^{\infty}\left(\sum_{k=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k} P_{n-k}^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x) P_{m-k}^{\left(\lambda_r+\frac{m_{\lambda_r}}{2}\right)}(x)\right) t_\alpha^n S_\alpha^m$$

Tsehaye K. Araaya [12]. Shows the following result.

Proposition8. For each$\lambda_r \in \mathbb{R}$, the corresponding orthogonal polynomial system with respect to the associated inner product satisfies the following relation.

$$\left(P_n^{(\lambda_r)}(x), P_n^{(\lambda_r)}(x)\right)_{\lambda_r,0} = \begin{cases} \sum_{k,r=0}^{n}\binom{m_{\lambda_r}}{k}\frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & if\ n < m_{\lambda_r} \\ \sum_{k,r=0}^{m_{\lambda_r}}\binom{m_{\lambda_r}}{k}\frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & if\ n \geq m_{\lambda_r} \end{cases}$$

Where $\mu = \lambda_r + \frac{m_{\lambda_r}}{2}$.

Proof: Using the relation (24) where $m = n$, we obtain

$$\left(P_n^{(\lambda_r)}(x), P_n^{(\lambda_r)}(x)\right)_{\lambda_r,0} =$$

$$\begin{cases} \sum_{k,r=0}^{n} \binom{m_{\lambda_r}}{k} \left(P_{n-k}^{(\mu)}(x), P_{n-k}^{(\mu)}(x)\right)_{\mu,0}, & if\ n < m_{\lambda_r} \\ \sum_{k,r=0}^{m_{\lambda_r}} \binom{m_{\lambda_r}}{k} \left(P_{n-k}^{(\mu)}(x), P_{n-k}^{(\mu)}(x)\right)_{\mu,0}, & if\ n \geq m_{\lambda_r} \end{cases}$$

$$= \begin{cases} \sum_{k,r=0}^{n} \binom{m_{\lambda_r}}{k} \frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & if\ n < m_{\lambda_r} \\ \sum_{k,r=0}^{m_{\lambda_r}} \binom{m_{\lambda_r}}{k} \frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & if\ n \geq m_{\lambda_r} \end{cases}$$

Since $\left(P_{n-k}^{(\mu)}(x), P_{n-k}^{(\mu)}(x)\right)_{\mu,0} = 0$, whenever $k > n > 0$, and

$$\left(P_n^{(\mu)}(x), P_n^{(\mu)}(x)\right)_{\mu,0} = \frac{2^{1-2\mu}\Gamma(n+2\mu)}{n!}$$

Remark 9. If $\lambda_r > 0$, the result in Theorem 4 and Proposition 8 hold true, for then $m_{\lambda_r} = 0$.

Corollary10. In the case when $2\lambda_r \leq 0$ is an integer, we have the following result:

$$\left(P_n^{(\lambda_r)}(x), P_n^{(\lambda_r)}(x)\right)_{\lambda_r,0} = \begin{cases} \sum_{k,r=0}^{n} \binom{m_{\lambda_r}}{k}, & if\ n < m_{\lambda_r} \\ \sum_{k,r=0}^{\infty} \binom{m_{\lambda_r}}{k} = 2^{m_{\lambda_r}}, & if\ n \geq m_{\lambda_r} \end{cases}$$

We give our following results.

Corollary 11. If $\lambda_r \geq 0, \mathbb{N}_{\lambda_r}$ has a least element equal to zero, where $r = 1,2,3, \ldots$, then we have

(i) $$\left\| P_n^{(\lambda_r)}(x) \right\|_{\lambda_r,0}^2 = \frac{2^{1-2\lambda_r}\Gamma(n+2\lambda_r)}{n!}$$

(ii) $\left\{P_n^{(\lambda_r)}(x)\right\}$ is orthonormal and $\Gamma(n + 2\lambda_r)$

$$= \frac{n!}{2^{1-2\lambda_r}}, \text{Hence}$$

$$\Gamma(n) = \frac{n!}{2} \quad \text{for} \quad n \in \mathbb{N}$$

(iii) $\left\| \sum_{\alpha=0}^{\infty} \sum_{r=1}^{\infty} G_{\lambda_r}(x, t_\alpha) \right\|_{\lambda_r,0} \leq$

$$\sum_{\alpha=0}^{\infty} \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{2^{1-2\lambda_r}\Gamma(n+2\lambda_r)}{n!} \|t_\alpha^m\|_{\lambda_r,0}$$

Proof:

(i) For $\mu = \lambda_r + \frac{m_{\lambda_r}}{2}$ and when $m_{\lambda_r} = 0$, we have $\mu = \lambda_r$ then $\left\|P_n^{(\lambda_r)}(x)\right\|_{\lambda_r,0}^2 = \left(P_n^{(\lambda_r)}(x), P_n^{(\lambda_r)}(x)\right)_{\lambda_r,0} = \frac{2^{1-2\lambda_r}\Gamma(n+2\lambda_r)}{n!}$

(ii) If $m_{\lambda_r} = 0$ Corollary 10, implies that $\left\|P_n^{(\lambda_r)}(x)\right\|_{\lambda_r,0} = \sum_{k,r=0}^{n}\binom{m_{\lambda_r}}{k} = 1$

and (i) gives that

$\Gamma(n + 2\lambda_r) = \frac{n!}{2^{1-2\lambda_r}}$,

and for $\lambda_r = 0$ then $\Gamma(n) = \frac{n!}{2}$.

(iii) We can show that $\left\|\sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty} G_{\lambda_r}(x, t_\alpha)\right\|_{\lambda_r,0} = \sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\|P_n^{(\lambda_r)}(x) t_\alpha^m\right\|_{\lambda_r,0}$

$$\leq \sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\left\|P_n^{(\lambda_r)}(x)\right\|_{\lambda_r,0}\|t_\alpha^m\|_{\lambda_r,0}$$

$$\leq \sum_{\alpha=0}^{\infty}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\frac{2^{1-2\lambda_r}\Gamma(n + 2\lambda_r)}{n!}\|t_\alpha^m\|_{\lambda_r,0}.$$

Now we conclude that (ii) above determined orthonormality of $P_n^{(\lambda_r)}(x)$, at the least mimimum element.

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