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Symmetries and Exact Solutions of the Wave Equation on Torus.

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Abstract:

This paper introduces wave equation and focus on investigating its Lie symmetries and exact solutions. We Determine the Lie symmetries of wave equation and we use different symmetries to obtain similarity reductions of wave equation and exact solutions. Our motivation is to extend some of the ideas developed in this paper to investigate wave equation on a surface with variable Gaussian curvature, specifically the torus.

Keywords: Wave equation, symmetry, Invariant, subalgebra.

Introduction:

Most physical processes ranging from natural to engineering sciences involve instantaneous changes through a series of states. This often makes it possible for the scientists to often explicitly express such physical processes in terms of differential mathematical models. In addition to the classical methods, recent literature provides a series of new methods to analyze such differential models and this had made the analysis of such physical process easier. Some of the geometrically rich and physically significant classes of surfaces include the ruled surfaces, which consist of straight lines, surfaces of revolution, which result from a rotation of a plane curve around an axis, tubular surfaces, which are defined by space curves, as well as minimal surfaces, which are important in the theory of soap films. These classes of surfaces possess a wide range of applications, for example, in computer graphics, digital design,

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architecture, engineering design, study of biological membranes, sheet metal based industries, the study of key objects in most nonlinear phenomena in physics and field theories etc. Surfaces of revolution form a large class of surfaces, which are generated by rotating a plane curve about an axis. Hence, such surfaces naturally posses nice symmetry properties. This makes surfaces of revolutions and related problems an interesting area of contemporary research, and particularly of importance in the fields of physics, engineering, computer graphics and other disciplines involving models of physical processes with a natural symmetries. Well known examples of surfaces of revolution include cylinder, cone, sphere, hyperboloid, ellipsoid, Gabriel's horn, Pseudosphere, torus, catenoid and tractoid. A common phenomena that appears in many fields like fluid mechanics, plasma physics, hydrodynamics, and general relativity is the wave phenomena. Therefore, the studies related to exact solutions and properties of wave equations have remained of significant interest. In differential geometry of surfaces, one of the fundamental concepts investigated is the Gaussian curvature, first studied in depth by Carl Friedrich Gauss (1825-1827), who showed that curvature was an intrinsic property of a surface. Gaussian curvature makes the torus different from other surfaces of revolution. The sphere has positive curvature, hyperboloid of one sheet has negative curvature whereas for a plane, the curvature is zero. On the other hand, depending on where the point is situated on a torus, the curvature can be positive, negative or zero.

1. Wave Equation:

We use the definition of the Laplace operator from differential geometry, it can easily be shown that the wave equation on torus with the metric

$$g = ds^2 = dx^2 + (1+\cos x)^2 dy^2$$

takes the form.

$$u_{tt} = -\frac{\sin x}{1+\cos x}u_x + u_{xx} + \frac{1}{(1+\cos x)^2}u_{yy} \tag{1.1}$$

The main tool for our investigation is Lie symmetry method which is a method of studying differential equations using their symmetries and was introduced by Sophus Lie. Lie's classical approach is based on finding a symmetry group associated with the differential equation. This is a local Lie group of point transformations on the space of independent and dependent variables of differential equation that maps solutions to solutions. The classical method of Lie allows computing the symmetry group associated to a given differential equation. This symmetry group can further be used for many important applications in the context of differential equations. For instance, for determination of similarity solutions, for reduction of order of ODEs, for reduction of the number of variables in PDEs. Hence, Lie symmetry method is a powerful general method for analyzing PDEs and can be efficiently employed to study those problems that have an implicit or explicit symmetry. Since the modern treatment of the classical Lie symmetry theory by Ovsiannikov, the theory of symmetries of differential equations has been studied intensely and has substantially grown. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available.

Symmetries of the Wave Equation on Torus

Lie symmetry analysis of differential equations was initiated by the Norwegian mathematician Sophus Lie (1842-1899). Today, this area of research is actively engaged. The method of determing the classical Lie symmetries of partial differential equation is systematic which is described in many books. To obtain the Lie symmetries of the PDE (1.1), we consider a one parameter Lie group infinitesimal transformation in (x, y, t, u) given by:

$$x^* = x + \varepsilon\xi(x, y, t, u) + 0(\varepsilon^2)$$

$$y^* = y + \varepsilon\vartheta(x, y, t, u) + 0(\varepsilon^2)$$

$$t^* = t + \varepsilon\tau(x, y, t, u) + 0(\varepsilon^2)$$

$$u^* = u + \varepsilon\varphi(x, y, t, u) + 0(\varepsilon^2)$$

where ε is the group parameter, hence the corresponding infinitesimal generator of the Lie algebra is of the form

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \vartheta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}$$

to obtain information on how the partial derivatives of u in the PDE (1.1) are transformed, we need up to second prolongation of X i.e. $X^{[2]}$. Using the invariance condition

$$X^{[2]}F\Big|_{F=0} = 0,$$

$$F = F(x,y,t,u,u_x,u_{xx},u_{yy},u_{tt}) = u_{tt} + \frac{\sin x}{1+\cos x}u_x - u_{xx} - \frac{1}{(1+\cos x)^2}u_{yy}$$

Comparison of the coefficients of u and its derivatives yields the following system of 13 determing equations.

$$e_1 : \xi_u = 0$$

$$e_2 : \vartheta_u = 0$$

$$e_3 : \tau_u = 0$$

$$e_4 : \varphi_{uu} = 0$$

$$e_5 : \xi_t - \tau_x = 0$$

$$e_6 : \xi_x - \tau_t = 0$$

$$e_7 : \vartheta_t - \frac{1}{(1+\cos x)^2}\tau_y = 0$$

$$e_8 : \vartheta_x + \frac{1}{(1+\cos x)^2}\xi_y = 0$$

$$e_9 : -\frac{\sin x}{1+\cos x}\xi - \xi_x + \vartheta_y = 0$$

$$e_{10} : \varphi_{tt} + \frac{\sin x}{1+\cos x}\varphi_x - \varphi_{xx} - \frac{1}{(1+\cos x)^2}\varphi_{yy} = 0$$

$$e_{11} : -\frac{\sin x}{1+\cos x}\tau_x + \frac{1}{(1+\cos x)^2}\tau_{yy} + 2\varphi_{ut} = 0$$

$$e_{12} : \xi f_{xx} - \frac{\sin x}{1+\cos x}\xi_x + 2\varphi_{xu} - \frac{1}{(1+\cos x)^2}\xi_{yy} = 0$$

$$e_{13} : \vartheta_{tt} - \vartheta_{xx} + \frac{\sin x}{1+\cos x}\vartheta_x + 2(\varphi_{uy} - \vartheta_{yy})\frac{1}{(1+\cos x)^2} = 0$$

The major tool for solving such a system of linear partial differential equations is triangulation similar to Gaussian elimination.

Using $(e_6)_x - (e_5)_t$ and $(e_6)_t - (e_5)_x$ we obtain the following:

$$e_{14} : \tau_{xx} - \tau_{tt} = 0$$

$$e_{15} : \xi_{xx} - \xi_{tt} = 0$$

By $e_{13} - (e_7)_t + (e_8)_x - \frac{\sin x}{1+\cos x}e_8 + \frac{1}{(1+\cos x)^2}(e_9)_y,$ we note that

$$e_{16} : \xi_{xy} + 2\varphi_{uy} = 0$$

By $e_{11} - (e_9)_t + (e_7)_y$ and $e_{12} - (e_9)_x + (e_8)_y$ respectively we observe that

$$e_{17} : \xi_{tx} + 2\varphi_{ut} = 0$$

$$e_{18} : \xi_{xx} + 2\varphi_{ux} = 0$$

By $2(e_{10})_u-(e_{17})_t-\dfrac{\sin x}{1+\cos x}e_{18}+(e_{18})_x+\dfrac{1}{(1+\cos x)^2}(e_{16})_y,$ we note that

$$e_{19}:-\frac{\sin x}{1+\cos x}\xi_{xx}+\frac{1}{(1+\cos x)^2}\xi_{xyy}=0$$

By $((e_7)_y-(e_9)_t)_t-\dfrac{1}{(1+\cos x)^2}(e_6)_{yy}+e_{19},$ we note that

$e_{20}:\xi_{ttx}=0$ and $\xi_{xxx}=0$ by e_{15}

By $((e_9)_x-(e_8)_y)_x+e_{19}-\dfrac{2\sin x}{1+\cos x}((e_9)_x-(e_8)_y),$ we note that

$$e_{21}:\xi\sin x-2\cos x(1+\cos x)\xi_x=0$$

Differentiating e_{21} twice with respect to t and using $e_{20},\ e_{15}$ and e_{19} respectively, we note that

$$\xi_{tt}=0\Rightarrow\xi_{xx}=0\Rightarrow\xi_{xyy}=0\Rightarrow\xi=p(y)xt+g(y)x+h(y)t+i(y)$$

It then follows immediately from the substitution of ξ in e_{21} that

$$p(y)=g(y)=h(y)=i(y)=0\Rightarrow\xi=0$$

By $e_5,\ e_6\ (e_7)_y$ and $e_8,\ e_9,\ (e_7)_t$ we respectively observe that

$$\tau=ay+k_1 \text{ and } \vartheta=bt+k_2$$

Substituting the above in e_7 indicates $a=b=0$

Similarly by $e_4,\ e_{18},\ e_{16}$ and e_{17} we note that

$$\varphi=k_3u+f(x,y,t)$$

Where $f(x,y,t)$ is a function satisfying the equation (1.1).

Hence the associated symmetry algebra of equation (1.1) is spanned by:

$$X_1=\frac{\partial}{\partial y},\ X_2=\frac{\partial}{\partial t},\ X_3=u\frac{\partial}{\partial u}\text{ and }X_f=f\frac{\partial}{\partial u}$$

Table 1: Commutator table for the Lie algebra

	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

3. Symmetry Reductions and Invariant Solutions:

In this section we carry out symmetry reductions and then determine some analytical solutions of the wave equation (1.1) obtained by solving the reduced PDEs. The symmetry reductions will be performed using the standard method of the introduction of similarity variables. The reduced equations would be tried for solution through some process or other techniques.

3.1 Reduction by 1-Dimensional Subalgebra:

Under this subsection, we consider one of the cases resulting from 1- dimensional subalgebra whose corresponding similarity variables are obtained. These similarity variables are then used to deduce the reduced PDE and the form of the solution of the equation (1.1).

Consider the subalgebra $\mathrm{L}_1 = \langle X_1 + aX_2 \rangle$

The similarity variables are

$$\xi_1 = x, \quad \xi_2 = y - \frac{t}{a} \quad \text{and} \quad V(\xi_1, \xi_2) = u$$

Substitution of similarity variable of the equation (1.1) is of the form

$$V(\xi_1, \xi_2) = u$$

Where $V(\xi_1, \xi_2)$ satisfies the following reduced PDE in 2 independent variables

$$\frac{1}{a^2}\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} - \frac{\sin \xi_1}{1+\cos \xi_1}\frac{\partial V}{\partial \xi_1} + \frac{1}{(1+\cos \xi_1)^2}\frac{\partial^2 V}{\partial \xi_2^2} \qquad (3.1)$$

If we let $V(\xi_1,\xi_2)=g(\xi_1)+h(\xi_2)$ is a solution of the PDE (3.1) above, then the functions $g(\xi_1)$ and $h(\xi_2)$ respectively satisfy the equation below

$$g''(\xi_1)-\frac{\sin\xi_1}{1+\cos\xi_1}g'(\xi_1)=c\left(\frac{1}{a^2}-\frac{1}{(1+\cos\xi_1)^2}\right)$$

$$h''(\xi_2)=0$$

The substitution of $r=\tan\left(\frac{\xi_1}{2}\right)$ reduces the first ODE to form below

$$g''(r)=c\left(\frac{4}{a^2(1+r^2)^2}-1\right)$$

This implies that

$$g(r)=cr\left(\frac{2}{a^2}\tan^{-1}r-\frac{1}{2}r+k_1\right)+k_2$$

For the second ODE, we have

$$h(\xi_2)=\frac{c}{2}(\xi_2+k_3)^2+k_4$$

Thus by back substitution of the similarity variables, we note that the solution of the equation (1.1)

$$u(x,y,t)=\frac{c}{2}\left\{\left(\frac{2x}{a^2}-\tan\left(\frac{x}{2}\right)+k_1\right)\tan\left(\frac{x}{2}\right)+(y-\frac{t}{a}+k_3)^2\right\}+k$$

3.2 Reduction by 2-dimensional subalgebra.

In this subsection we reduce the order the equation (1.1) using a two dimensional subalgebra. This allows us to write the equation (1.1) as an ODE. We also give the corresponding exact solution.

Consider the subalgebra $\mathrm{L}_2=\langle X_1,aX_2+bX_3\rangle$

By the first symmetry X_1 whose similarity variables are of the form

$$\xi_1=x,\quad \xi_2=t \text{ and } V(\xi_1,\xi_2)=u$$

We note that the solution of the equation (1.1) is of the form

$$u = V(\xi_1, \xi_2)$$

Where $V(\xi_1, \xi_2)$ satisfies the following PDE in 2- independent variables

$$\frac{\partial^2 V}{\partial \xi_2^2} = \frac{\partial^2 V}{\partial \xi_1^2} - \frac{\sin \xi_1}{1 + \cos \xi_1} \frac{\partial V}{\partial \xi_1} \tag{3.2}$$

Since the two symmetries X_1 and $aX_2 + bX_3$ commute i.e. $[X_1, aX_2 + bX_3] = 0,$ second symmetry $aX_2 + bX_3$ is inherited by PDE above by the theorem on P-285 of [10]. Hence

$$Y = a\frac{\partial}{\partial \xi_2} + V\frac{\partial}{\partial V}$$

is a symmetry of the PDE (3.2). Its similarity variables are of the form

$r(\xi_1, \xi_2) = \xi_1$ and $w(r) = \ln V - \frac{b}{a}\xi_2$

These reduces the PDE (3.2) to second order ODE below

$$w''(r) + w'(r)^2 - \tan\left(\frac{r}{2}\right)w'(r) = k^2, \quad k = \frac{b}{a}$$

The substitution of $w(r) = \ln(z(r)\sec(r/2))$ in the ODE reduces it to

$$z''(r) - (k^2 + 1/4)z(r) = 0$$

This implies that

$$z(r) = \begin{cases} k_1 \cos mr + k_2 \sin mr, & -m^2 = (k^2 - 1/4) \\ k_1 r + k_1, & k^2 = 1/4 \\ k_1 e^{mr} + k_2 e^{-mr}, & m^2 = (k^2 - 1/4) \end{cases}$$

It follows by back substitution implies that the solution of the equation (1.1) takes the form

$$u(x,y,t)=\begin{cases} e^{kt}\sec(x/2)(k_1\cos mx+k_2\sin mx), & -m^2=(k^2-1/4) \\ e^{kt}\sec(x/2)(k_1x+k_2), & k^2=1/4 \\ e^{kt}\sec(x/2)(k_1e^{mx}+k_2e^{-mx}), & m^2=(k^2-1/4) \end{cases}$$

4. Conclusion:

As already clarified that a common phenomenon that appears in many fields like fluid mechanics, plasma physics, hydrodynamics, and general relativity is the wave phenomena. Therefore, the studies related to exact solutions and properties of wave equations are of a significant interest. However, exact solutions of wave equation on different surfaces are sometimes hard to find. In this paper we choose the surface torus which is a surface with rich geometric properties and consider wave equation. By implementing Lie's classical method we find symmetry algebra. Using carefully chosen similarity transformations, we succeed in finding two new solutions of wave equation on torus in terms of exponential and trigonometric functions.

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