

Weighted Strictly Singular Operators of Sequences on Invariant Subspaces

Emran Sasi AlThbet⁽¹⁾ and Mohammed Ali Bashir⁽²⁾

Abstract:

We investigate the twistorial derivations of self-dual string equation, the Maxwell equation and the Bogomolny equation via dimensional reductions of the equations of motion of the six-dimensional self-dual three-form field strength $H = dB$.

1. Introduction:

The complexified space-time \mathbb{C}^4 is used to replace Minkowski space-time $\mathbb{R}^{1,3}$ by the space of projective light cones in this space, $\mathbb{R}^{1,3} \times S^2$ which is diffeomorphic to the open subset $\mathbb{P}_0^3 \equiv \mathbb{P}^3/\mathbb{P}_1$ called Twistor Space. This is due to Penrose who further used the above correspondence to construct penrose transform by which we can describe solution of zero-rest mass fields by complex analytic data (cohomology groups). Moreover, holomorphic vector bundles over the Twistor space may also be used to describe Yang-Mills instantons on \mathbb{C}^4 . Then a twistorial description of chiral theories is demonstrated by a twistor space P^6 that enables a description of chiral zero-rest-mass fields on six-dimensional space-time and certain cohomology groups on P^6 . This construction generalizes Penrose-ward transform to P^6 .

In this review paper we introduce the dimensional reductions of the six-dimensional spinor fields to four and three space-time dimensions. The twistor space P^6 would contain the ambitwistor space that describes Maxwell and Yang-Mills equations, the hyper-plane twistor space that describes the self-dual string

(1) Academy of Engineering Sciences, Khartoum, Sudan.

(2) Academy of Engineering Sciences, Khartoum, Sudan.

equation and the minitwistor space for monopole construction. In this paper we only describe Maxwell and Yang-Mills fields in addition to self-dual strings. So we consider the reduction of P^6 to ambitwistor and hyper-plane twistor spaces.

2.Reduction of M^6 to M^4 :

To dimensionally reduce M^6 to M^4 , we split the $6 \cong 4 \wedge 2$ representation of $so(6, \mathbb{C}) \cong sl(4, \mathbb{C})$ into the bi-fundamental representation $(\mathbf{2}, \mathbf{2})$ of $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \cong so(4, \mathbb{C})$ plus twice the trivial representation to obtain

$$x^{AB} \rightarrow (x^{\alpha\dot{\beta}}, \varepsilon^{\alpha\beta} x^+, \varepsilon^{\dot{\alpha}\dot{\beta}} x^-), \quad (1)$$

where $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots = 1, 2$. The symplectic forms $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ of $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ can be used to raise and lower $sl(2, \mathbb{C})$ spinor indices. Four-dimensional space-time M^4 is then given by the quotient $M^4 := M^6/D_{M^6}$ with the distribution $D_{M^6} := \langle \partial_{\pm} \rangle$.

A general three-form $H = dB$ in six dimensions is given by a pair of symmetric bi-spinors $H_{AB} = \partial_C (A B_B)^C$ and $H^{AB} = \partial^{C(A} B_C^{B)}$ via a (traceless) two-form potential B_B^A . Self-duality of H is equivalent to saying that $H^{(AB)} = 0$.

A two-form potential in six dimensions B_B^A reduces then to

$$B_B^A \rightarrow (A_{\alpha\dot{\alpha}}^+, A_{\alpha\dot{\alpha}}^-, B_{\alpha\beta} = B_{(\alpha\beta)}, B_{\dot{\alpha}\dot{\beta}} = B_{(\dot{\alpha}\dot{\beta})} = \emptyset) \quad (2)$$

and represents in four dimensions two one-form potentials $A_{\alpha\dot{\alpha}}^{\pm}$, a two-form potential $(B_{\alpha\beta}, B_{\dot{\alpha}\dot{\beta}})$ and a scalar field \emptyset .

Correspondingly, gauge transformations of B_B^A ,

$$B_B^A \rightarrow B_B^A + \partial^{AC} \Lambda_{CB} - \partial_{BC} \Lambda^{CA}, \quad (3)$$

where $\Lambda_{AB} = -\Lambda_{BA}$, reduce in four dimensions to $\Lambda_{AB} \rightarrow (\Lambda_{\alpha\dot{\alpha}}, \Lambda^+, \Lambda^-)$ with

$$A_{\alpha\dot{\alpha}}^{\pm} \mapsto A_{\alpha\dot{\alpha}}^{\pm} + \partial_{\alpha\dot{\alpha}} \Lambda^{\pm},$$

$$B_{\alpha\beta} \mapsto B_{\alpha\beta} + \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{(\alpha\dot{\alpha}} \Lambda_{\beta)\dot{\beta}}, \quad (4)$$

$$B_{\dot{\alpha}\dot{\beta}} \mapsto B_{\dot{\alpha}\dot{\beta}} + \varepsilon^{\alpha\beta} \partial_{\alpha(\dot{\alpha}} \Lambda_{\beta)\dot{\beta}}$$

$$\emptyset \mapsto \emptyset.$$

Furthermore, the (first-order) self-duality equation $H^{AB} = \partial^{C(A} B_C^{B)}$ reduces to

$$H_{\alpha\dot{\alpha}} := \varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} \beta_{\dot{\alpha}\dot{\gamma}} - \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} B_{\alpha\gamma} = \partial_{\alpha\dot{\alpha}} \emptyset \quad (5-a)$$

and

$$f_{\dot{\alpha}\dot{\beta}}(A^+) = 0 \quad \text{and} \quad f_{\alpha\beta}(A^-) = 0, \quad (5-b)$$

where $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ are the self-dual and anti-self-dual parts of the curvature of a potential $A_{\alpha\dot{\alpha}}$, i.e.

$$f_{\alpha\beta}(A) := \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{(\alpha\dot{\alpha}} \Lambda_{\beta)\dot{\beta}} \quad \text{and} \quad f_{\dot{\alpha}\dot{\beta}}(A) := \varepsilon^{\alpha\beta} \partial_{\alpha(\dot{\alpha}} \Lambda_{\beta)\dot{\beta}} \quad (5-c)$$

Note that under this decomposition, $H_{AB} \rightarrow (f_{\alpha\beta}(A^+), f_{\dot{\alpha}\dot{\beta}}(A^-), \partial_{\alpha\dot{\alpha}}\phi)$. Equation (5-a) is the self-dual string equation $H = \star_4 d\phi$ in spinor notation, while (5-b) says that the curvature of A^+ (respectively, A^-) is self-dual (respectively, anti-self-dual).

3. Maxwell Equation:

The Maxwell equation for a gauge potential $A_{\alpha\dot{\alpha}}$ in spinor notation is given by:

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} + \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma} = 0, \quad (6-a)$$

And the Bianchi identity is given by:

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} - \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma} = 0, \quad (6-b)$$

so that the equations for $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ decouple as

$$\varepsilon^{\dot{\beta}\dot{\gamma}} \partial_{\alpha\dot{\beta}} f_{\dot{\alpha}\dot{\gamma}} = 0 = \varepsilon^{\beta\gamma} \partial_{\beta\dot{\alpha}} f_{\alpha\gamma}.$$

The general Maxwell field $A_{\alpha\dot{\alpha}}$ is expressed as $A_{\alpha\dot{\alpha}} = A_{\alpha\dot{\alpha}}^+ + A_{\alpha\dot{\alpha}}^-$, where $A_{\alpha\dot{\alpha}}^\pm$ reflects the degrees of freedom.

4. Bogomolny Equation (in three dimensions):

To further reduce to three dimensions, we split the $(\mathbf{2}, \mathbf{2})$ of $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ into the $\mathbf{3} \oplus \mathbf{1}$ of $sl(2, \mathbb{C})$: $x^{\alpha\dot{\alpha}} \rightarrow (x^{\alpha\beta} = x^{(\alpha\beta)}, x^{[12]})$. We then have $M^3 := M^4/D_{M^4}$ with $D_{M^4} := \langle \frac{\partial}{\partial x^{[12]}} \rangle$. Here, the field strength of a gauge potential reduces directly according to:

$$\partial_{\alpha\beta} A_{\gamma\delta} - \partial_{\gamma\delta} A_{\alpha\beta} = \varepsilon_{\alpha\gamma} f_{\beta\delta} + \varepsilon_{\beta\delta} f_{\alpha\gamma}, \quad (7)$$

and the BPS subsector of the reduced Maxwell equation is described by the Abelian Bogomolny equation $F = \star_3 d\phi$, which reads in spinor notation as

$$f_{\alpha\beta} = \partial_{\alpha\beta} \phi \quad (8)$$

This equation can be obtained from the self-dual string equation by defining $= \frac{\partial}{\partial x^{[12]}} \lrcorner H$. In spinor notation, this amounts to defining $f_{\alpha\beta} = H_{\alpha\beta}$ and (5-a) reduces to (8).

5. Ambitwistor space:

Recall that three-dimensional totally null-planes in M^6 , which are in one-to-one correspondence with points in P^6 , are given by

$$x^{AB} = x_0^{AB} + \varepsilon^{ABCD} \mu_C \lambda_D, \tag{9}$$

where μ_A is defined modulo terms proportional to λ_A . In view of the splitting $sl(4, \mathbb{C})$ into $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ used in reducing M^6 to M^4 , we now split the spinors λ_A and μ_A according to $\lambda_A \rightarrow (\mu_\alpha, \lambda_{\dot{\alpha}})$ and $\mu_A \rightarrow (\kappa_\alpha, v_{\dot{\alpha}})$. (10)

Correspondingly, upon using (1), the equation (9) decomposes as

$$x^{\alpha\dot{\beta}} = x_0^{\alpha\dot{\beta}} + \mu^\alpha v^{\dot{\alpha}} - \kappa^\alpha \lambda^{\dot{\alpha}}, x^+ = x_0^+ + v_{\dot{\alpha}} \lambda^{\dot{\alpha}}, \text{ and } x^- = x_0^- + \kappa_\alpha \mu^\alpha, \tag{11}$$

where we have raised spinor indices with $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$, respectively. Next we impose $x^\pm = 0 = x_0^\pm$ to dimensionally reduce to four dimensions. Hence, $v_{\dot{\alpha}} \lambda^{\dot{\alpha}} = 0$ and $\kappa_\alpha \mu^\alpha = 0$. There are essentially three cases emerging from these equations:

i. $\mu_\alpha = 0$ but $\lambda_{\dot{\alpha}} \neq 0$ and κ_α arbitrary and $v_{\dot{\alpha}} \propto \lambda_{\dot{\alpha}}$ so that the first equation (11) reduces to $x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} - \kappa^\alpha \lambda^{\dot{\alpha}}$. (12)

This equation parametrises anti-self-dual two-planes in four dimensions (so-called α -planes).

ii. $\mu_\alpha \neq 0$ but $\lambda_{\dot{\alpha}} = 0$ and $\kappa_\alpha \propto \mu_\alpha$ and $v_{\dot{\alpha}}$ arbitrary so that $x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \mu^\alpha v^{\dot{\alpha}}$, (13)

which parametrises self-dual two-planes in four dimensions (so-called β -planes).

iii. we can have both $\mu_\alpha \neq 0$ and $\lambda_{\dot{\alpha}} \neq 0$ together with $\kappa_\alpha \propto \mu_\alpha$ and $v_{\dot{\alpha}} \propto \lambda_{\dot{\alpha}}$. This gives $x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \varrho \mu^\alpha \lambda^{\dot{\alpha}}$ for $\varrho \in \mathbb{C}$, (14)

which are the null-lines in four dimensions arising from intersecting α -planes and β -planes.

Choosing the α - or β -planes will lead to either Penrose's twistor space or Penrose's dual twistor space which are the twistor spaces parametrising such null-planes. In the following,

we shall work with the null-lines obtained from intersections of α - and β -planes. This yields the ambitwistor space P^5 which is the twistor space parametrising such null-lines. As we shall show momentarily, the ambitwistor space P^5 as well as the correspondence space F^6 can be obtained by factoring out certain distributions. The quadric equation defining P^6 in the open subset \mathbb{P}_0^7 of \mathbb{P}^7 will reduce to the quadric equation defining P^5 in the open subset $\mathbb{P}^3 \times \mathbb{P}^3$ of $\mathbb{P}^3 \times \mathbb{P}^3$.

We now decompose all the twistor coordinates (z^A, λ_A) on P^6 according to the splitting of $sl(4, \mathbb{C})$ into $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ to obtain

$$(z^A, \lambda_A) \rightarrow (z^\alpha, -\omega^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}). \quad (15)$$

Let us first consider the base space \mathbb{P}^3 of the fibration $P^6 \rightarrow \mathbb{P}^3$. To reduce \mathbb{P}^3 with homogeneous coordinates λ_A to $\mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $(\mu_\alpha, \lambda_{\dot{\alpha}})$, we consider the corresponding reduction of the structure sheaf $\mathcal{O}_{\mathbb{P}^3}$ of \mathbb{P}^3 . Local sections f of $\mathcal{O}_{\mathbb{P}^3}$ fulfil the equation $Yf = 0$, where $Y := \lambda_A \frac{\partial}{\partial \lambda_A}$ is the Euler vector field on \mathbb{P}^3 .

This reflects the invariance under re-scalings $\lambda_A \rightarrow t\lambda_A$, with $t \in \mathbb{C}^*$. Local sections f of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ fulfil $\mu_\alpha \frac{\partial}{\partial \mu_\alpha} f = 0$ and $\lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} f = 0$. Therefore, the

$$\begin{aligned} &\text{quotient of } P^3 \text{ by the distribution} \\ D_-(P^3) := &\langle \mu_\alpha \frac{\partial}{\partial \mu_\alpha} - \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} \rangle \quad (16) \\ &\text{can be identified with } P^1 \times P^1. \end{aligned}$$

Analogously, one reduces P^7 with homogeneous coordinates (z^A, λ_A) to $P^3 \times P^3$. Since we are interested in non-compact versions, let us directly remove the P^3 defined by $z^A \lambda_A \neq 0$ and $\lambda_A = 0$ from P^7 to obtain the ambient space $P_{-0}^7 = P^7 / P^3 \cong \mathcal{O}_-(P^3) \oplus \mathbb{C}^4$ of the twistor space P^6 . The quotient of P_{-0}^7 by the distribution

$$D_-(P_{-0}^7) := \langle z^\alpha \frac{\partial}{\partial z^\alpha} + \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} - \omega^{\dot{\alpha}} \frac{\partial}{\partial \omega^{\dot{\alpha}}} - \mu^\alpha \frac{\partial}{\partial \mu^\alpha} \rangle \quad (17)$$

can be identified with $\mathbb{P}^3 \times \mathbb{P}^3$, where \mathbb{P}^3 and \mathbb{P}^3 are each bi-holomorphic to the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathbb{C}^2$. The quadric condition $z^A \lambda_A = 0$, which defines $P^6 \rightarrow \mathbb{P}_0^7$, descends to the quadric equation

$$z^\alpha \mu_\alpha - \omega^{\dot{\alpha}} \lambda_{\dot{\alpha}} = 0, \tag{18}$$

which defines the ambitwistor space $P^5 \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ as a quadric hypersurface of $\mathbb{P}^3 \times \mathbb{P}^3$. Note that \mathbb{P}^3 is Penrose's twistor-space of four-dimensional space time while \mathbb{P}^3 is the dual twistor space.

The correspondence space F^6 is obtained as the quotient of $F^6 \cong \mathbb{C}^6 \times \mathbb{P}^3$ by the distribution

$$D_{F^6} := \left\langle \frac{\partial}{\partial x^\pm}, \mu^\alpha \frac{\partial}{\partial \mu^\alpha} - \lambda_{\dot{\alpha}} \frac{\partial}{\partial \lambda_{\dot{\alpha}}} \right\rangle, \tag{19}$$

and we have $F^6 := F^6/D_{F^6} \cong \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$. Altogether, we arrive at the following double fibration:

$$\begin{array}{ccc} & F^6 & \\ \pi_3 \swarrow & & \searrow \pi_4 \\ P^5 & & M^4 \end{array} \tag{20}$$

where π_4 is the trivial projection and

$$\pi_3: (x^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}}, \mu_\alpha) \mapsto (z^\alpha, \omega^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) = (x^{\alpha\dot{\alpha}} \lambda_{\dot{\alpha}}, x^{\alpha\dot{\alpha}} \mu_\alpha, \mu_\alpha, \lambda_{\dot{\alpha}}). \tag{21}$$

Note that the twistor distribution in this case is of rank one and generated by the vector field $\mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$, i.e. $P^5 \cong F^6 / \langle \mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \rangle$ with

$$\partial^{\alpha\dot{\alpha}} := \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x^{\beta\dot{\beta}}}.$$

Geometrically, a point x in four-dimensional space-time M^4 corresponds to a holo-morphic embedding of $\hat{x} := \pi_3(x_4^{-1}(x)) \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow P^5$. On the other hand, a point p in ambitwistor space P^5 corresponds to a null line $\pi_4(x_3^{-1}(x)) \rightarrow M^4$ given by

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \varrho \mu^\alpha \lambda^{\dot{\alpha}} \text{ with } \varrho \in \mathbb{C}, \tag{22}$$

in agreement with our initial choice of null-space.

Moreover, if we introduce the two projections $pr_{1,2}: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ to the first and second copy of \mathbb{P}^1 , respectively, and in addition

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l) := pr_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(l), \tag{23}$$

$$\Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^p(k, l) := \Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^p(pr_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(l))$$

for $k, l \in \mathbb{Z}$, we get the following sequence for the ambitwistor space

$$0 \rightarrow P^5 \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1) \otimes \mathbb{C}^2 \xrightarrow{k} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \rightarrow 0. \tag{24}$$

Here, $k: (z^\alpha, \omega^{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) \mapsto z^\alpha \mu_\alpha - \omega^{\dot{\alpha}} \lambda_{\dot{\alpha}}$. Upon dualising and twisting by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ the Euler sequence for $\mathbb{P}^1 \times \mathbb{P}^1$, we find

$$0 \rightarrow \Omega_{\mathbb{P}^1 \times \mathbb{P}^1}^p(1,1) \rightarrow P^5 \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1). \quad (25)$$

This implies that P^5 can be identified with the bundle of first-order jets $\text{Jet}^1 \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ as a consequence of the jet-sequence

$$0 \rightarrow \Omega_X^1(S) \rightarrow \text{Jet}^1 S \rightarrow S \rightarrow 0 \quad (26)$$

for an Abelian sheaf S on a complex manifold X (see e.g. [12]).

The above constructions show that we have a factorisation of the tangent bundle T_{M^4} into the two bundles of undotted and dotted chiral spinors. We shall denote these bundles by S and \tilde{S} and therefore, $T_{M^4} \cong S \otimes \mathcal{O}_{M^4} \tilde{S}$, which is the reduction of the corresponding factorisation in six dimensions. Note that such a factorisation amounts to choosing a holomorphic conformal structure. re, we shall make use of the following notation (for $k, l \in \mathbb{Z}$):

6. Penrose—Ward Transforms:

Let us consider an open set $U \subset M^4$ and define $\hat{U} := \pi_3(\pi_4^{-1}(U)) \subset P^5$ with covering $\hat{\mathcal{U}} = \{\hat{U}_\alpha\}$. We start from holomorphic line bundles over \hat{U} which are holomorphically trivial on any $\hat{x} \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \hat{U}$. Such line bundles are characterised by Cech one-cocycles $\hat{f} = \{\hat{f}_{ab}\} \in H^1(\hat{U}, \mathcal{O}_{\hat{U}})$. The pull-back of \hat{f} to the correspondence space can be split holomorphically, $f'_{ab} = \pi_3^* \hat{f}_{ab} = h'_a - h'_b$. Since \hat{f}_{ab} gets annihilated by the twistor distribution, we find $A' := \mu_\alpha \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} h'_a$ which is globally defined. Hence A' must be of the form $A' := \mu_\alpha \lambda_{\dot{\alpha}} A^{\alpha\dot{\alpha}}$, where $A_{\alpha\dot{\alpha}}$ depends only on space-time. Since the twistor distribution is one-dimensional, we do not obtain any spacetime field equations for $A_{\alpha\dot{\alpha}}$. Moreover, since the splitting $f'_{ab} = h'_a - h'_b$ is not unique, we can always consider $h'_a \mapsto h'_a + \varphi'$, where φ' is defined globally on $\hat{U}' := \pi_4^{-1}(U) \subset P^5$. Therefore, φ' can only depend on space-time (since the \mathbb{P}^1 s are compact) and thus, it corresponds to transformations of the form $A_{\alpha\dot{\alpha}} \mapsto A_{\alpha\dot{\alpha}} + \partial^{\alpha\dot{\alpha}} \varphi'$. In summary, this shows that $H^1(\hat{U}, \mathcal{O}_{\hat{U}})$ can be identified with the Maxwell potentials on U modulo gauge

transformations. Notice that this construction also applies to the non-Abelian setting, that is, to Yang-Mills potentials.

As is well-known, in order to construct self-dual (or anti-self-dual) solutions to the Maxwell/Yang-Mills equations, one employs Ward's construction [5] starting from holomorphic vector bundles over Penrose's twistor space \mathbb{P}^3 (or the dual twistor space $\tilde{\mathbb{P}}^3$) subject to certain triviality conditions. Because ambitwistor space incorporates both twistors and dual twistors, it can be used to give a twistor interpretation of the Maxwell/Yang-Mills equations. As we have seen above, however, the ambitwistor space itself is not quite sufficient to recover these equations. To resolve this problem, one needs to thicken the ambitwistor space into its ambient space $\mathbb{P}^3 \times \tilde{\mathbb{P}}^3$ to a certain order.

Theorem 2. ([8,9]) *Let U be an open subset of M^4 such that any null line intersects U in a convex set. Then there is a one-to-one correspondence between gauge equivalence classes of complex holomorphic solutions to the Yang-Mills equations on U and equivalence classes of holomorphic vector bundles which are holomorphically trivial on any $\hat{x} \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow P^5$ for all $x \in U$ and which admit an extension to a 3-rd order thickening $P_{[3]}^5$ of P^5 in $\mathbb{P}^3 \times \tilde{\mathbb{P}}^3$.*

Note that if the holomorphic vector bundle can be extended to a finite neighbourhood within the ambient space $\mathbb{P}^3 \times \tilde{\mathbb{P}}^3$, then the space-time gauge field constructed from this vector bundle is either self-dual, anti-self-dual or Abelian. Thus, if one is only interested in the Maxwell equation (as we are in the present case) one may work with holomorphic line bundles on the ambient space $\mathbb{P}^3 \times \tilde{\mathbb{P}}^3$ which are holomorphically trivial on $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3 \times \tilde{\mathbb{P}}^3$. Since F^3 is Penrose's twistor space while \tilde{F}^3 its dual, one finds a self-dual and an anti-self dual field strength. Both can be linearly superposed to obtain a solution to the Maxwell equation. This is possible as the equations for the two helicities decouple.

7. Hyperplane Twistors and Self-Dual Strings:

In this section, we introduce a new twistor space P^3 . We shall see below, this twistor space underlies a Penrose-Ward transform mapping a certain cohomology group on P^3 to

solutions to the self-dual string equation on M^4 in a bijective manner.

While ambitwistor space describes the intersection of α - and β -planes, the hyperplane twistor space will describe the span of the union of both types of intersecting null-planes. Because these two kinds of planes intersect along a null-line, the span describes a three-dimensional hyperplane in M^4 , hence the name hyperplane twistor space.

In the corresponding double fibration, space-time obviously remains the same. Recall that the two spheres in the correspondence space $F^6 \cong \mathbb{C}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$ specify the choice of an α - and a β -plane. Because we need the same data in the definition of a hyperplane twistor, the correspondence space remains the same, too. The equivalence relation between points, however, is different: while two points in the correspondence space are equivalent if they correspond to the same null-line in the case of ambitwistors, in the case of hyperplane twistors, two points in the correspondence space are considered equivalent if they correspond to the same hyperplane. Therefore the twistor distribution for the hyperplane twistor space contains that of the ambitwistor space, but it is strictly larger, and the hyperplane twistor space is a subspace of the ambitwistor space.

Explicitly, the hyperplane twistor space P^3 can be obtained by quotienting P^5 by the distribution

$$D_{P^5} := \langle \mu^\alpha \frac{\partial}{\partial z^\alpha}, \lambda^{\dot{\alpha}} \frac{\partial}{\partial \omega^{\dot{\alpha}}} \rangle \quad (27)$$

It is rather straightforward to see that $P^3 := P^5/D_{P^5}$ is bi-holomorphic to the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Altogether, we may write down the following double fibration:

$$\begin{array}{ccc} & F^6 & \\ \pi_5 \swarrow & & \searrow \pi_6 \\ P^3 & & M^4 \end{array} \quad (28)$$

where π_6 is the trivial projection and

$$\pi_4: (x^{\alpha\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) \mapsto (z, \mu_\alpha, \lambda_{\dot{\alpha}}) = (x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}, \mu_\alpha, \lambda_{\dot{\alpha}}) \quad (29)$$

Note that the twistor distribution is of rank three and generated by the vector fields $\mu_\alpha \partial^{\alpha\dot{\alpha}}$ and $\lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$, i.e. $P^6 \cong F^4 / \langle \mu_\alpha \partial^{\alpha\dot{\alpha}}, \lambda_{\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \rangle$, with $\partial^{\alpha\dot{\alpha}} := \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x^{\beta\dot{\beta}}}$ as before.

The geometric twistor correspondence here is as follows. By virtue of the incidence relation $z = x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}$, a point $x \in M^4$ corresponds to a holomorphic embedding of $\hat{x} := \pi_5(\pi_6^{-1}(x)) \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow P^3$, while a point $p \in P^3$ corresponds to a hyperplane $\pi_6(\pi_5^{-1}(p)) \rightarrow M^4$ in space-time. To see this, note that the incidence relation $z = x^{\alpha\dot{\alpha}} \mu_\alpha \lambda_{\dot{\alpha}}$ can be solved for fixed $p = (z, \mu, \lambda) \in P^3$ by

$$x^{\alpha\dot{\alpha}} = x_0^{\alpha\dot{\alpha}} + \mu^\alpha v^{\dot{\alpha}} - \kappa^\alpha \lambda^{\dot{\alpha}} \tag{30}$$

Here, $x_0^{\alpha\dot{\alpha}}$ is a particular solution and κ_α and $v_{\dot{\alpha}}$ are arbitrary, which parametrise translations of $x_0^{\alpha\dot{\alpha}}$ along totally null two-planes (the α - and β -planes). The apparent four parameters in the spinors $v^{\dot{\alpha}}$ and κ^α are reduced to three, because the shifts

$$\kappa^\alpha \mapsto \kappa^\alpha + \varrho \mu_\alpha \quad \text{and} \quad v_{\dot{\alpha}} \mapsto v_{\dot{\alpha}} + \varrho \lambda_{\dot{\alpha}} \quad \text{for } \varrho \in \mathbb{C} \tag{31}$$

leave the solution (32) invariant.

Remark:

The above constructions show again that we have a factorisation of the tangent bundle T_{M^4} into the two bundles of undotted and dotted chiral spinors. As before, we shall denote these bundles by S and \tilde{S} and therefore, $T_{M^4} \cong S \otimes \mathcal{O}_{M^4} \tilde{S}$. We introduce

$$\mathcal{O}_{P^3}(k, l) := pr^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l) \quad \text{for } k, l \in \mathbb{Z}, \tag{32}$$

where pr is the bundle projection $pr: P^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, l)$ was defined in (23).

References:

- [1] D. Popov and C. Saemann, On supertwistors, the Penrose-Ward transform and $N = 4$ super Yang-Mills theory, *Adv. Theor. Math. Phys.* 9 (2005) 931 [hep-th/0405123].
- [2] [BL'05] The geometric structure of Lorentzian manifolds with twistor spinors in low dimension. In: *Dirac Operators - Yesterday and Today*, eds: J.P.Bourguignon, T.Bransen, A.Chamseddine, O.Hijazi, R.Stanton; 229- 240, International Press 2005.
- [3] Saemann, On the mini-superambitwistor space and $N = 8$ super Yang-Mills theory, *Adv. Math. Phys.* 2009 (2009) 784215 [hep-th/0508137].
- [4] Saemann, R. Wimmer, and M. Wolf, A twistor description of six-dimensional $N = (1,1)$ super Yang-Mills theory, *JHEP* 1205 (2012) 020 [1201.6285 [hep-th]].
- [5] Devchand, Integrability on light-like lines in six-dimensional superspace, *Z. Phys. C* 32 (1986) 233.
- [6] Christain Samann and Martinwolf, On twistors and conformal field theories from six dimensions. Arxiv: 1111.2539v3, Jan 2013.
- [7] Dzhunushaliev and H.-J. Schmidt [DS'00] New vacuum solutions of conformal Weyl gravity, *J. Math. Phys.* 41 (2000) 3007-3015.
- [8] Witten, Twistor-like transform in ten dimensions, *Nucl. Phys. B* 266 (1986) 245.
- [9] [Fs'06] Causal conformal vector fields, and singularities of twistor spinors. Preprint 2006.
- [10] H.Baum & I.Kath [BK'99] Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds. *Ann.Global Anal. Geom.* 17 (1999) 1-17.
- [11] H.Baum & F.Leitner [BL'04] The twistor equation in Lorentzian spin geometry. *Math. Z.* 247 (2004), 795812.
- [12] J.H.Herranz & M.Santander [HS'02] Conformal symmetries of spacetimes. *J. Phys. A: Math Gen.* 35 (2002), 66016618.
- [13] [Li'06] Conformal Einstein spaces in N -dimensions. II. *J. Geom. Phys.* 56 (2006), 386-404
- [14] L. J. Mason and D. Skinner, An ambitwistor Yang-Mills Lagrangian, *Phys. Lett. B* 636 (2006) 60 [hep-th/0510262].

- [15] M.Blau, J.Figueroa-O'Farrill and G.Papadopoulos [BF'02] Penrose limits, super-gravity and brave dynamics. *Class.Quantum Grav.* 19(2002)4753-4805.
- [16] M.Blau, M.Borunda, M.O'Loughlin & G.Papadopoulos [BP'04] Penrose limits and spacetime singularities. *Class.Quantum Grav.* 21 (2004) L43-L49.
- [17] M. Schottenloher [So'97] A mathematical introduction to conformal field theory. Springer 1997 U.
- [18] N. Berkovits and S. A. Cherkis, Higher-dimensional twistor transforms using pure spinors, *JHEP* 0412 (2004) 049 [hep-th/0409243].
- [19] R. S. Ward and R. O. Wells, Twistor geometry and field theory, Cambridge University Press, Cambridge, 1990.
- [20] R. S. Ward, Completely solvable gauge field equations in dimension greater than four, *Nucl. Phys. B* 236 (1984) 381.
- [21] R. Pool, Some applications of complex geometry to field theory, PhD Thesis, Rice University Texas, 1981.
- [22] T. N. Bailey and M. G. Eastwood, Complex paraconformal manifolds - Their differential geometry, *Forum Math.* 3 (1991) 61.