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On Eigenfunction Expansion Solution for Plane CouetteFlows of Second Grade Fluid with Fractional Calculus

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Abstract:

Ivan [2] suggesteda modifying separationof variables (i.e., eigenfunction - expansions), in this paper we investigatethe solutions obtained by this new approach, we find that it produces the correct solution shown to agree identically with the fractional calculus approach obtainedfor a class of unsteady flows for the generalized second grade fluid with the fractional derivative model between two parallel plates.

1.Introduction:

In recent work, Ivan Christovexplainedthatin[2]most mathematics and engineering textbooks describe the process of the steady stateof a linear parabolic partial differential equation as a technique for obtaining a boundary-value problem with homogeneous boundary conditions that can be solved by separation of variables(i.e., eigenfunction expansions).

While this method produces the correct solution for the start-up of the flow ofa Newtonian fluid between parallel plates, it can lead to erroneous solutions to the corresponding problem for a class of non-Newtonian fluids. He showed that the reason for this is the non-rigorous enforcement of the start-upconditions in the

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textbook approach, whichleadtoa violation of the principle of causality. Nevertheless, these boundary-value problems can be solved correctly using eigenfunction expansions, and he presented the formulation that makes this possible (in essence, an application of Duhamel's principle)[3]. The solutions obtained by this new approach are shown to agree identically with those obtained by using the Laplace transform in time only, a technique that enforces the proper start-up condition implicitly. In this paper we examinethe suggestion of Ivan's paper for the generalized second grade fluid with the fractional dervetive in the paper of Tan[8].

The paper is organized as follows: In section 3, wegave asummary of the solutions of generalized second gradefluid by fractional dervetivewhich were given by Tan[8] for the flow between two parallel plates with one starting to move suddenly and the other at rest, and the flow between two rigid boundaries which are suddenly started. In Section 4, the solutions are derived by using the textbook eigenfunction expansion technique modified by Ivan [2],then insection 5 we gave a critical discussion ofthe results in 3and 4.

2. BASIC EGUATIONS:

We consider the flow of a second grade non-Newtonian fluid between two horizontal parallel impermeable plates.The distance between two plates is d.The general constitutive equation given by Rivlin and Ericksen [4] can be written in the following form [1]:

$$\tau = -\rho I + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \qquad (1)$$

Where$p$is the pressure, I is the unit tensor, μ is the dynamic viscosityα_1, α_2, are the normal stress moduli,A_1 ,A_2 are first two Rivlin-Ericksen kinematic tensors Where:

$$A_1 = \nabla V + (\nabla V)^T \qquad (2)$$

$$A_2 = \frac{dA_1}{dt} + A_1(\nabla V) + A_2(\nabla V)^T \qquad (3)$$

Where$(dA1 / dt)$ denotes the material time derivative, V is the velocity field and grad is the gradient operator, T is the matrix transpose.

If the second-grade fluid given by (1) is compatible with thermodynamics, then thematerial moduli must meet the following restriction (Dunn and Rajagopal) [7].

$$\mu \geq 0, \alpha_1 \geq 0 \quad and \quad \alpha_1 + \alpha_2 = 0 \tag{4}$$

consider unidirectional flows of the form

$$V = u(y,t)i \tag{5}$$

Where u is the velocity in the x-coordinate direction and i the unit vector in the x-direction. Onsubstituting the expression (1) for the stress T in the balance of linear momentum.

$$divT + \rho b + \rho dv\ /dt \tag{6}$$

using (3) and the fact that the fluid is incompressible, one obtains

$$\mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} - \rho\left(\frac{\partial u}{\partial t}\right) = 0 \tag{7}$$

where ρ is the density of the fluid. The above equation impliesthe existence of the following flows in the case of a second grade fluid.

For fractional calculus a direct approach to creat fractional rheological constitutive equations is to replace the regular time derivative of ordinary RCE by fractional time derivatives of non-intger order [2].The fractional RCE of a generalized second grade fluid is obtained by rewriting Eq(3) as:

$$A_2 = D_t^{\beta} A_2 + (\nabla V)A_1 + A_1(\nabla V) + A_1(\nabla V)^T \tag{8}$$

where D_t^{β} is Riemann-Liouvill definition[8]:

$$D_t^{\beta} = \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{t} (t-\tau)^{-\beta} f(\tau) d\tau, \ 0 < \beta < 1 \tag{9}$$

Where $\Gamma(.)$ is the Gamma function ,D_t^{β} denot the fractionaltime derivatives of non-intger order. Equation (1), (2) and (9) are called the fractional RCE of generalized second grade fluid. When $\beta = 1$, (5) may be simplified as (3), and when $\alpha_1 = 0$, the constitutive equation of the generalized second grade fluid can be simplified to those of classical Newtonian viscous fluid. In the absence of the body forces the equation of motion is:

$$\rho \frac{DV}{Dt} = \nabla.\tau \tag{10}$$

from (5), (9) we can obtain

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 D_t^{\beta}\left[\frac{\partial^2 u}{\partial y^2}\right] \tag{11}$$

This equation is the governing equation for the flow of the generalized second grade fluid between two parallel plates.

3. Solution by fractional calculus of generalized second grade fluid Tan[8]:

3.1: The flow between two parallel plates with one starting to move suddenly and the other at rest:

Tan Wenchang[8], used fractional calculus approach to solve the governing equation for the flow of the second grade fluid between two parallel plates.

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 D_t^{\beta} \left[\frac{\partial^2 u}{\partial y^2}\right]$$

We suppose that the distance between the parallel plats is d. The plates at $y = 0$ is initially at rest and it brought suddenly to the steady velocity$U$. The plate at $y = d$ is always at rest. Then the boundary and initial condition are:

$$u(0,t) = U(t) = U\begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{12}$$

$$u(d,t) = 0 \tag{13}$$

$$u(y,t) = 0 \text{ for } 0 < y < d \quad t \leq 0 \tag{14}$$

He introduceda transformation function substituting into the governing equationand becomes:

$$\rho \left(\frac{\partial v}{\partial t}\right) = \mu \frac{\partial^2 v}{\partial y^2} + \alpha_1 D_t^{\beta} \left[\frac{\partial^2 v}{\partial y^2}\right] \tag{15}$$

under conditions:

$$v(0,t) = 0 \tag{16}$$

$$v(d,t) = 0 \tag{17}$$

$$v(y,0) = -U\, A_{ss}(y), \quad 0 < y < d, t \leq 0, \; A_{ss}(y) = \frac{(y-d)}{d} \tag{18}$$

and seek a solution by the form

$$v(y,t) = \sum_{k=1}^{\infty} T_k(t) \Psi_k(y) \tag{19}$$

have the Mittage –Leffer function

$$\int_0^{\infty} e^{-st}\, t^{\alpha n+\lambda-1} E_{\alpha,\lambda}^{n}(-at^{\alpha})dt = \frac{n! s^{\alpha-\lambda}}{(s^{\alpha}+a)^{n+1}} \tag{20}$$

Where:

$$E_{\alpha,\lambda}^{n}(z) = \frac{d^n}{dz} E_{\alpha,\lambda}^{n}(z) = \sum_{j=0}^{\infty} \frac{(j+n)! z^{j}}{j!\Gamma(\alpha_{j+}\alpha_n+\lambda)} \tag{21}$$

Mittage –Leffer function [5]

$$T(t) = -\frac{2U}{k\pi}\sum_{k=0}^{\infty} \frac{(-1)^n}{n!}\; c_k^n . t^n_{1-\beta,1+n\beta}\; \left(-b_k t^{1-\beta}\right) \qquad (22)$$

arrived at:

and the result of Tan [8] for second grade fluid by fractional calculus approach is:

$$U(y,t) = H(t)\left[U\left(1-\frac{y}{d}\right) - 2U\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{sin\left(\frac{k\pi y}{d}\right)}{k\pi} . \frac{(-1)^n}{n!}\; c_k^n . t^n\, E^{(n)}_{1-\beta,1+n\beta}\; \left(-b_k t^{1-\beta}\right)\right] \qquad (23)$$

Where:

$$b_k = \frac{\alpha_1 k^2\pi^2}{\rho d^2},\; c_k = \frac{\mu k^2\pi^2}{\rho d^2}$$

3.2:The flow between two rigid boundaries which are suddenly started:

Let the fluid be bounded by two parallel boundaries and be initially at rest. The fluid motion is due to the plate at y = 0 being brought suddenly to the steady velocity U, with the plate at y = d as free surface (du/dy),then the boundary conditions are:

$$u(0,t) = U\begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad (24)$$

$$\frac{du}{dy} = 0 \qquad (25)$$

$$u(y,t) = 0\,, for \quad 0 < y < d \quad t \leq 0 \qquad (26)$$

and using the same method as that in section 3.1, we have:

$$U(y,t) = U - \frac{4U}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{sin\left((2k+1)\pi y\,/2d\right)}{2k+1} . \frac{(-1)^n}{n!}\; c_k^n . t^n\, E^{(n)}_{1-\beta,1+n\beta}\; \left(-b_k t^{1-\beta}\right) \qquad (27)$$

Where:

$$b_k = \frac{\alpha_1 (k+1)^2\pi^2}{\rho d^2},\; c_k = \frac{\mu (k+1)^2\pi^2}{\rho d^2}$$

4- Resolution using the modifiyngeigenfunctionexpansion suggestedby Ivans [2]

4.1:The flow between two parallel plates with one starting to move suddenly and the other at rest:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 D_t^{\beta} \left[\frac{\partial^2 u}{\partial y^2}\right]$$

We suppose that the distance between the parallel plates is d. The plate at y = 0 is initially at rest and it brought suddenly to the steady velocity. The plate at y = d is always at rest. Then the boundary and initial conditionsare:

$$u(0,t) = UH(t) = U\begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \tag{28}$$

$$u(y,0) = 0,\ u(d,t) = 0 \tag{29}$$

$$u(y,t) = 0 \qquad for \quad 0 < y < d \quad t \leq 0, \tag{30}$$

Now, we introduce a transformation function:

$$v(y,t) = u(y,t) - U\,A_{ss}(y)H(t), A_{ss}(y) = \left(1 - \frac{y}{d}\right) \tag{31}$$

Noting that: $\partial(H(t))A_{ss} / \partial t = \delta(t)A_{ss}$, thenIBVP becomes:

$$\rho\left(\frac{\partial v}{\partial t} + UA_{ss}(y)\delta(t)\right) = \mu \frac{\partial^2 v}{\partial y^2} + \alpha_1 D_t^{\beta} \left[\frac{\partial^2 v}{\partial y^2}\right] \tag{32}$$

Under conditions:

$$v(0,t) = 0 \tag{33}$$

$$v(d,t) = 0 \tag{34}$$

$$v(\ y,0) = 0 \tag{35}$$

From [2] the initial condition, being understood as the state prior tostartup, $\lim_{t\to 0-} u(y,t)$, naturally remains zero. This interpretation is a demonstration of Duhamel’s principle [3](Duhamel, 1833; Bartels and Churchill, 1942; Sneddon, 2006), namely, that a time-varying boundary condition can be “exchanged” for a homogeneous boundary condition at the “cost” of adding a time-varying source term to the linear BVP. Notice that the textbook approach exchanges the inhomogeneous boundary conditions for a homogeneous boundary condition at the cost of an inhomogeneous initial condition. Philosophically, this is already problematic because the cumulative effects of the boundary condition from t = 0 up to t =∞ , have been “condensed” into an initial condition and imposed t = 0, an act that readily violates the principle of causality, namely “no outputbefore the input” (Toll, 1956)[9].

As before, the method of separation of variables suggests that the ansatzsubstituting the latter into Eq. (38) and using the orthogonality relation from Eq. (30), we seek a solution of the form:

$$v(y,t) = \sum_{k=1}^{\infty} T_k(t)\Psi_k(y) \tag{36}$$

So we have:

$$\frac{\partial v}{\partial t} = \sum_{k=1}^{\infty} \frac{dT_k}{dt}\Psi_k(y)\ ,\ \frac{\partial^2 v}{\partial y^2} = \sum_{k=1}^{\infty} T_k(t)\Psi''_k(y)$$

And so:

$$\rho\left(\sum_{k=1}^{\infty} \frac{dT_k}{dt}\Psi_k(y) + UA_{ss}(y)\delta(t)\right) = \mu\left(\sum_{k=1}^{\infty} T_k(t)\Psi''_k(y)\right) + \alpha_1\left[\sum_{k=1}^{\infty} D_t^{\beta}[T_k(t)]\Psi''_k(y)\right] \tag{37}$$

Until we find that:

$$\frac{dT_k}{dt} + b_k D_t^{\beta}\left(T_k(t)\right) + c_k\left(T_k(t)\right) = -\frac{2U}{k\pi}\delta(t) \tag{38}$$

$$T_k(0) = 0$$

Soby using Laplacetransform:

$$T(s) = -\frac{2U}{k\pi}.\frac{1}{c_k}.\frac{c_k s^{-\beta}}{\left(S+b_k s^{\beta}+c_k\right)}\left(1+\frac{c_k s^{-\beta}}{\left(b_k+s^{1-\beta}\right)}\right)^{-1} = -\frac{2U}{k\pi}\sum_{n=0}^{\infty}(-1)^n\frac{s^{-n\beta-\beta}}{\left(s^{1-\beta}+b_k\right)^{n+1}} \tag{39}$$

$$T(t) = -\frac{2U}{k\pi}\sum_{k=0}^{\infty}\frac{(-1)^n}{n!}\,c_k^n.t^n\,E_{1-\beta,1+n\beta}^{n}\left(-b_k t^{1-\beta}\right) \tag{40}$$

$$U(y,t) = H(t)\left[U\left(1-\frac{y}{d}\right) - 2U\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{sin\left(\frac{k\pi y}{d}\right)}{k\pi}.\frac{(-1)^n}{n!}\,c_k^n.t^n\,E_{1-\beta,1+n\beta}^{(n)}\left(-b_k t^{1-\beta}\right)\right] \tag{41}$$

4.2: The flow between two rigid boundaries which are suddenly started:

Suppose that the fluid is bounded by two rigid boundaries at y = 0 and y = d are initially at rest and brought suddenly to the steady velocities U_0 and U_1 , respectively the governing differential equation is:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 D_t^{\beta}\left[\frac{\partial^2 u}{\partial y^2}\right]$$

$$u(0,t) = U\begin{cases} 0\ ,\ t < 0 \\ 1\ ,\ t \geq 0 \end{cases} \tag{42}$$

With boundary conditions:

$$u(0,t) = U_0 \tag{43}$$

$$u(d,t) = U_1 \tag{44}$$

$$u(y,0) = 0 \mathit{for} \quad 0 < y < d \quad t \leq 0 \tag{45}$$

Now, we introduce a transformation function:

$$v(y,t) = u(y,t) - UH(t) \tag{46}$$

Substituting into Eq. (42)~(46), we find that :

$$\rho\left(\frac{\partial v}{\partial t} + U\delta(t)\right) = \mu \frac{\partial^2 v}{\partial y^2} + \alpha_1 D_t^{\beta}\left[\frac{\partial^2 v}{\partial y^2}\right] \tag{47}$$

Under conditions:

$$v(0,t) = 0 \tag{48}$$

$$\frac{\partial v}{\partial y}(d,t) = 0 \tag{49}$$

$$v(y,t) = 0 \mathit{for} \quad 0 < y < d \quad t \leq 0 \tag{50}$$

and using the same method as that in section 4.1, we have:

$$T(t) = -\frac{2U}{(2k+1)\pi}\sum_{k=0}^{\infty} \frac{(-1)^n}{n!}\ c_k^n . t_{1-\beta,1+n\beta}^n\ \left(-b_k t^{1-\beta}\right) \tag{51}$$

So we have:

$$U(y,t) = H(t)\left[U - \frac{4U}{\pi}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{\sin\left(\frac{k\pi y}{d}\right)}{(2k+1)} . \frac{(-1)^n}{n!}\ c_k^n . t_{1-\beta,1+n\beta}^n\ \left(-b_k t^{1-\beta}\right)\right] \tag{52}$$

4. Discussion and Conclusion:

Ivan Christovexplainedthat in [2]most mathematics and engineering textbooks describe the process of steady state of a linear parabolic partial differential equation as a technique for obtaining a boundary-value problem with homogeneous boundary conditions that can be solved by separation of variables (i.e ., eigenfunction- expansions), these boundary-value problems can be solved correctly using eigenfunction expansions, and he presented the formulation that makes this possible so, in this paper we examined the suggestion ofIvan's for the generalized second grade fluid with the fractional dervetive in the paper of Tan [8]. Clearly we found that in section 3 using the fractional calculus approach obtainedfor a class of unsteady flows for the generalized second grade fluid specifically Eqs. (23) and (27) agrees exactly with the solution of modifying separation of variables (i.e., eigenfunction expansions) in section 4inEqs. (41) and (52) are identicalso, the modifying separation ofvariables (i.e., eigenfunction expansions) is accurate and more flexible than theordinary second grade fluid model on describing the properties of a viscoelastic fluid.

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