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Symplectic Geometry

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Abstract:

In this paper we show that a dynamical system may be described geometrically by a structure (symplectic) defined on a bundle space. The structure is in fact invariant under the symplectic group.

1 – Introduction

Mechanics has passed through several progresses. First started with Newtonian mechanics motivated by Aristotelian philosophy about motion, force, space and time. Then a historical achievement has taken place due to the great effort of the mathematician Euler followed by his colleague Lagrange, who together introduced the generalized coordinates to form mechanics based on the calculus of variation. In fact during this era mechanics has developed mathematics quite a lot, particularly mathematical analysis.

Another important and equivalent formulation of mechanics was done by Hamilton. In fact via Legendre transformation the domain of definition has been transformed to position coordinates and momentum coordinates (q , p).

Now came an important and significant formulation of mechanics using a geometrical language. Starting with Cartan and his followers mechanics became geometrical using what is known as exterior calculus where differential forms and its algebra has been greatly utilized. In this paper we shall illustrate this motion, using the modern concepts of exterior calculus.

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In fact Euler Lagrange equations will be shown to be equivalent to Hamilton's equations. The latter will also be equivalent to a geometrical form which is free of coordinates and global.

2 – The classical Lagrangian and Hamiltonian Mechanics

2 – 1 Lagrangian formulation with constraints

For a physical system in which all of the forces (except forces of constraints) are derivable from a scalar potential, the motion of the system between times t_1 and t_2 is such that the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt \qquad (1)$$

Where L = T – V, has a stationary value for the actual path of the motion.

Applying the calculus of variations and integrating by parts, one obtains Lagrange's equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad (2)$$

For holonomic constraints, that is constraints connecting the coordinates of the particles and the time expressible in the form[9,10]

$F(t, x) = 0$,

We modify the action to be

$$S = \int_{t_1}^{t_2} \left\{ L + \sum_{\alpha=1}^{m} \lambda_\alpha F_\alpha \right\} dt \qquad (3)$$

Where the F_α are the **m** equations of constraint. Varying the coordinates q_i and the undetermined multipliers leads to a modified form of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}+\sum_{\alpha=1}^{m}\lambda_\alpha \frac{\partial F_\alpha}{\partial q_i}=0 \tag{4}$$

2 – 2 Hamiltonian formulation

We begin by defining the conjugate momenta to each coordinate q_i by

$$p_i=\frac{\partial L}{\partial q_i} \tag{1}$$

We express the Lagrangian in terms of P_i and q_i which will become our new canonical variables. The Hamiltonian is then generated by a simple Legendre transformation

$$H(q,p,t)=\dot{q}_i p_i-L(q,\dot{q},t) \tag{2}$$

This has the differential

$$dH=\dot{q}_i dp_i-\dot{p}_i dq_i-\frac{\partial L}{\partial t}dt \tag{3}$$

Which can also be written as

$$dH=\frac{\partial H}{\partial q_i}dq_i+\frac{\partial H}{\partial p_i}dp_i+\frac{\partial H}{\partial t}dt \tag{4}$$

From these we make the identification:

$$\dot{q}_i=\frac{\partial H}{\partial p_i}$$

$$-\dot{p}_i=\frac{\partial H}{\partial q_i} \tag{5}$$

$$-\frac{\partial L}{\partial t}=\frac{\partial H}{\partial t},$$

Which are known as Hamilton's equations of motion.

3 – Differential forms and Exterior calculus

(3 – 1) Definition

Let **M** be an n–dimensional differential manifold and let $(x^i)_p$, i = 1, … , n be a local coordinate system around **P** in **M**.

A p–form **W** is an expression, written locally as:[4,3]

$$W = \sum_{j_{1<j_p}} W_{j_1 \cdots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p}$$

(3 – 2) Definition

The exterior product (wedge product) of a p–form **W** and a q–form μ is the mapping[1,9]

$$\wedge : \Omega_x^p X \Omega_x^p = \Omega_x^{p+q} : (W, \mu) \rightarrow W \wedge \mu \text{ ,}$$

The exterior product $W \wedge \mu$ being such that $\forall X_1, \cdots, X_{p+q} \in T_x M$

$$W \wedge \mu (X_1, \cdots, X_{p+q}) = \frac{1}{p!q!} \sum_{i_{1,\cdots,i_{p+q}}}$$

$$\sigma^{i_{1,\cdots,i_{p+q}}} W(X_{i_1}, \cdots, X_{i_p}) \mu(X_{i_{p+1}}, \cdots, X_{i_{p+q}})$$

(3 – 3) Definition

The Lie derivative of q–form **T** with respect to **X**, at point $\mathbf{X_o}$, is

$$L_{X_0} T = \lim_{t=0} \frac{1}{t}\left(d\phi_t^{-} T_{\phi_t} X_0 - T_{x_0}\right) = \frac{d}{dt} \phi_t^* T \Big|_{t=0}$$

Let ω be a differential form of degree **p** on **M**.

(3 – 4) Definition

If $W=\sum_{j_1<j_p}\frac{dW}{dx^K}dx^K\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_p}$, is a p–form, then the exterior derivative of **W** is a (P + 1) form defined by

$$dW=\sum_{j_1<j_p,j_K}\frac{dW_{j_1\cdots j_p}}{dx^K}dx^K\wedge dx^{j_K}\wedge dx^{i_1}\wedge\cdots\wedge dx^{j_p}$$

(3 – 5) Definition

The inner product of **W** by **X** is the linear mapping

$$i_x:\Omega^P(M)\to\Omega^{P-1}(M):W\to i_xW$$

Defined by the inner product i_xW of **W** and **X** such that

$$\forall X_{i_1}\cdots X_{i_{p-1}}\in x(M)$$

(the set of vector fields on **M**).

$$i_xW(X_1,\cdots,X_{p-1})=W(X,X_1,\cdots,X_{p-1}).$$

The expression of the inner product is equivalent to[10,1]

$$i_xW=W_{i_1\cdots i_p}X^{i_1}dx^{i_2}\wedge\cdots\wedge dx^{i_p}$$
$$-W_{i_1\cdots i_p}X^{i_2}dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_p}$$
$$+\cdots+(-1)^{p-1}\omega_{i_1\cdots i_p}X^{i_p}dx^{i_1}\wedge\cdots\wedge dx^{I_{P-1}}$$

4 – The geometric formulation of Hamiltonian Mechanics

4 – 1 Symplectic manifold

Let **M** be a differentiable manifold. A 2–form ω defined on $T_pM\times T_pM$ is called a symplectic form if it satisfies the following conditions:[5,2]

(i) ω is closed: $d\omega=0$

(ii) ω is non-degenerate: det $\lfloor \omega(e_i, e_j) \rfloor \neq 0$

For any basis vectors (e_i) in $T_p M$. The pair (M , ω) is called a symplectic manifold.

Let (M , ω) be a symplectic manifold.

(1) We call a vector field **X** Hamiltonian (respectively, locally Hamiltonian or just symplectic) if X| ω is exact (respectively, closed).

(2) An isotopy ϕ^t of diffeomorphism is called a symplectic (respectively, Hamiltonian), if its generating vector fields X_0 are symplectic **C** respectively, Hamiltonian).

We denote by symp (M , ω) (respectively, ham (M , ω) the subset of symplectic vector field (respectively, the subset of hamiltonian vector fields).[7,8]

Obviously we have the inclusions Ham (M , ω) $\subset$ symp (M , ω) $\subset$ X(M).

4 – 2 Hamiltonian Mechanics

Hamiltonian Mechanics is supposed to be equivalent to Lagrangian formulation for many degrees of freedom. It is a generalization of Newtonian mechanics. We know that, Hamiltonian system is described by two variables in (n) unknowns. Namely q (position) and p(momentum), where **H** is the Hamiltonian function and

$$q = (q_1, \cdots, q_n),$$

$$p = (p_1, \cdots, p_n) \qquad (1)$$

Hamiltonian's equations to be the following form

$$\dot{q}^i = \frac{dH}{dp^i},$$

$$-\dot{p}^i = \frac{dH}{dq^i} \qquad (2)$$

i = (1, …, n), need to be transformed to a geometrical form.[6,2]

In fact we prove the following proposition: The Hamiltonian equation (2) are equivalent to the equation n
X| ω = – dH

Proof

In many symplectic manifold (M , ω), the symplectic form defines an isomorphism:

$$T_P M \rightarrow T_P^* M : X \rightarrow -X \mid \omega = -2\omega(X,.)$$

For each Hamiltonian function **H**, there corresponds a Hamiltonian vector field X_H.

In fact the Hamiltonian vector field is described by Hamilton's equations, whose solutions are given by the Hamiltonian function.

We interpret the above correspondence by

$$\omega(X,.) = -dH$$

In local Darboux coordinates, one has

$$dH = \sum_{i=1}^{n} \frac{dH}{dq_K} dq_K + \frac{dH}{dP_K} dp_K \qquad \text{(i)}$$

$$X_H = \sum_{i=1}^{n} \frac{dH}{dq_K} \frac{d}{dp_K} - \frac{dH}{dp_K} \frac{d}{dq_K}, \qquad \text{(ii)}$$

$$\omega = \sum \underset{ij}{\omega} dq^i \wedge dp^j \qquad \text{(iii)}$$

Contracting ω as above with X_H in (ii) we get exactly Hamilton's equations:

$$\dot{q}^i = \frac{dH}{dp^i},$$

$$-\dot{p}^i = \frac{dH}{dq^i}.$$

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