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The Geometrical interpretation of Dirac equation.

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Abstract:

In this paper we consider the spin complex. The study leads to the treatment of Dirac wave function as a spinor field or a cross-section of the spin bundle. Then we utilize Atiyah-Singer index theorem to evaluate the index of the spin complex.

Introduction

The elliptic complexes had been studied by several authors [11], [5]. The importance of this study is related to the existence of solutions of partial differential equations of elliptic type. Of particular interest, is the spin complex related to the celebrated Dirac equation. For this reason we introduced a covariant form of this equation in Section 1 of this paper [6]. Then we devoted Section 2 to the spinor formulation of Dirac equation [12], where the four-component wave function is represented by Weyl representation. The spin complex has been given a more mundane treatment in Section 3, [5]. Here the Dirac operator is defined using the covariant derivative that involve the metric on the base manifold and the spin connection 1-forms. Then we considered the index of the spin complex in Section 4.

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1. Dirac Equation

1.1 Formulation of Dirac Equation

To solve the negative probability density problem of the Klein-Gordon equation, people were looking for an equation which is first order in $\partial / \partial t$.

Such an equation is found by Dirac.

The Dirac equation is an equation derived by Paul Dirac in 1928 that describes relativistic spin $-1/2$ particles (fermions). It is interesting as one of the first uses of spinor calculus in mathematical physics. Dirac began with the relativistic equation of total energy:

$$E^2 = p^2c^2 + m^2c^4 \tag{1}$$

As Schrödinger had done before him, Dirac then replaced p with its quantum mechanical operator, $\hat{p} \Rightarrow i\hbar\nabla$.

It is difficult to take the square root of

$$E^2 = -\hbar c^2\nabla^2 + m^2c^4 \tag{2}$$

for a single wave function. One can take the inspiration from E&M:

Maxwell's equation are first-order but combining them gives the second-order wave equations.

Imagining that ψ consists of N components ψ_l.

$$\frac{1}{c}\frac{\partial \psi_l}{\partial t} + \sum_{k=1}^{3}\sum_{n=1}^{N}\alpha_{l_n}^k \frac{\partial \psi_n}{\partial x^k} + \frac{imc}{\hbar}\sum_{n=1}^{N}\beta_{l_n}\psi_n = 0, \tag{3}$$

where $l = 1,2,...,N$, and $x^k = x, y, z,\ k = 1,2,3$.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{4}$$

and α^k, β are $N \times N$ matrices. Using the matrix notation, we can write the equations as

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \alpha.\nabla\psi + \frac{imc}{\hbar}\beta\psi = 0, \tag{5}$$

where $\alpha = \alpha^1\hat{x} + \alpha^2\hat{y} + \alpha^3\hat{z}$. N components of ψ describe a new degree of freedom just as the components of the Maxwell field describe the polarization of the light quantum. In the spin of the particle and ψ is called a spinor. We would like to have positive definite and conserved probability, $\rho = \psi^*\psi$, where ψ^* is the hermitian conjugate of ψ (so is a row matrix).

Taking the hermitian conjugate of equation (5),

$$\frac{1}{c}\frac{\partial \psi^*}{\partial t} + \nabla\psi^*.\alpha - \frac{imc}{\hbar}\beta^*\psi^* = 0. \tag{6}$$

Multiplying the above equation by ψ and then adding it to $\psi^* \times Eq.(5)$, we obtain

$$\frac{1}{c}\left(\psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right) + \nabla\psi^*.\,\alpha^*\psi + \psi^*\alpha.\nabla\psi + \frac{imc}{\hbar}\left(\psi^*\beta\psi - \psi^*\beta^*\psi\right) = 0. \tag{7}$$

The continuity equation

$$\frac{\partial}{\partial t}\left(\psi^*\psi\right) + \nabla.j = 0 \tag{8}$$

can be obtained if $\alpha^* = \alpha,\ \beta^* = \beta$, then

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\psi^*\psi\right) + \nabla.\,(\psi^*\alpha\psi) = 0 \tag{9}$$

with

$$j = c\psi^*\alpha\psi. \tag{10}$$

From equation(5) we can obtain the Hamiltonian,

$$H\psi = i\hbar\frac{\partial \psi}{\partial t} = \left(c\nabla.\frac{\hbar}{i}\nabla + \beta mc^2\right)\psi. \tag{11}$$

One can see that H is hermitian if α, β are hermitian.

To derive properties of α, β, we multiply equation(5) by the conjugate operator,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \alpha.\nabla - \frac{imc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \alpha.\nabla + \frac{imc}{\hbar}\beta\right)\psi = 0$$

$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \alpha^i\alpha^j\partial_i\partial_j + \frac{m^2c^2}{\hbar}\beta^2 - \frac{imc}{\hbar}\left(\beta\alpha^i + \alpha^i\beta\right)\partial_i\right]\psi = 0 \tag{12}$$

we can rewrite $\alpha^i \alpha^j \partial_i \partial_j$ as $\frac{1}{2}(\alpha^i \alpha^j + \alpha^j \alpha^i)\,\partial_i \partial_j$.

Since it's a relativistic system, the second-order equation should coincide with the Klein-Gordon equation. Therefore, we must have

$$\alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij} I \quad (13)$$

$$\beta\alpha^i + \alpha^i \beta = 0 \quad (14)$$

$$\beta^2 = I \quad (15)$$

Because

$$\beta\alpha^i = -\alpha^i \beta = (-I)\alpha^i \beta. \quad (16)$$

1.2 Covariant form of Dirac equation

Define

$$\gamma^0 = \beta,$$

$$\gamma^j = \beta\alpha^j,\ j = 1,2,3$$

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3), \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu \quad (17)$$

Multiply equation(5) by $i\beta$,

$$i\beta \times (\frac{1}{c}\frac{\partial}{\partial t} + \alpha.\nabla + \frac{imc}{\hbar}\beta)\psi = 0,$$

$$\Rightarrow \left(i\gamma^0 \frac{\partial}{\partial x^0} + i\gamma^j \frac{\partial}{\partial x^j} - \frac{mc}{\hbar} \right)\psi = \left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right)\psi = 0 \quad (18)$$

using the short-hand notation:

$$\gamma^\mu \partial_\mu \equiv \partial\!\!\!/,\ \gamma^\mu \mathrm{A}_\mu \equiv \mathrm{A}\!\!\!/,\ \left(i\partial\!\!\!/ - \frac{mc}{\hbar} \right)\psi = 0 \quad (19)$$

From the properties of the α^j and β matrices, we can derive

$$\gamma^{0t} = \gamma^0, \quad (hermitian) \quad (20)$$

$$\gamma^{jt} = (\beta\alpha^j)^* = \alpha^{j*}\beta^* = \alpha^j \beta = -\beta\alpha^j = -\gamma^j, \ (anti-hermitian) \quad (21)$$

$$\gamma^{\mu*} = \gamma^0 \gamma^\mu \gamma^0\ , \quad (22)$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I\ . \quad (clifford\ a\lg ebra) \quad (23)$$

Conjugate of the Dirac equation is given by

$$-i\partial_\mu \psi^* \gamma^{\mu*} - \frac{mc}{\hbar}\psi^* = 0$$

$$\Rightarrow -i\partial_\mu \psi^* \gamma^0 \gamma^\mu \gamma^0 - \frac{mc}{\hbar}\psi^* = 0 \qquad (24)$$

We will define the Dirac adjoint spinor $\overline{\psi}$ by $\overline{\psi} = \psi^* \gamma^0$.
Then

$$i\partial_\mu \overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0. \qquad (25)$$

The four-current is

$$\frac{j^\mu}{c} = \overline{\psi}\gamma^\mu \psi = \left(\rho, \frac{j}{c}\right), \partial_\mu j^\mu = 0. \qquad (26)$$

1.3 Properties of the γ^μ matrices

We may form new matrices by multiplying γ matrices together. Because different γ matrices anticommute, we only need to consider products of different γ's and the order is not important. We have 16 different matrices,

$$\begin{aligned} &I \\ &\gamma^0, i\gamma^1, i\gamma^2, i\gamma^3 \\ &\gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, i\gamma^2\gamma^3, i\gamma^3\gamma^1, i\gamma^1\gamma^2 \\ &i\gamma^0\gamma^2\gamma^3, i\gamma^0\gamma^3\gamma^1, i\gamma^0\gamma^1\gamma^2, \gamma^1\gamma^2\gamma^3 \\ &i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5 (= \gamma^5). \end{aligned} \qquad (27)$$

Denoting them by $\Gamma_l, l = 1,2,\ldots,16$, we can derive the following relations.

(a) $\Gamma_l\Gamma_m = a_{lm}\Gamma_n,\ a_{lm} = \pm 1 \ or \pm i\alpha$.

(b) $\Gamma_l\Gamma_m = I$ if and only if $l = m$.

(c) $\Gamma_l\Gamma_m = \pm\Gamma_m\Gamma_l$.

(d) If $\Gamma_l \neq I$, there always exists a Γ_k, such that $\Gamma_k\Gamma_l\Gamma_k = -\Gamma_l$.

(e) $Tr(\Gamma_l) = 0 \ for \ \Gamma_l \neq I$.

(f) Γ_l are linearly independent:

$\sum_{k=1}^{16} x_k\Gamma_k = 0$ only if $x_k = 0,\ k = 1,2,\ldots,16$.

(g) Corollary: any 4×4 matrix X can be written uniquely as linear combination of the

$\Gamma's$.

$$X=\sum_{k=1}^{16}x_k\Gamma_k$$

$$Tr\,(X\,\Gamma_m)=x_m Tr\,(\Gamma_m\Gamma_m)+\sum_{k\neq m}x_k Tr\,(\Gamma_k\Gamma_m)=x_m Tr\,(I)=4x_m$$

where $x_m=\frac{1}{4}Tr\,(x\Gamma_m)$.

(h) Any matrix X that commutes with γ^μ for all μ is a multiple of the identity.

(i) Pauli's fundamental theorem:

Given two sets of 4×4 matrices γ^μ and $\gamma^{'\mu}$ which both satisfy

$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}I$,

there exists a nonsingular matrix S such that

$\gamma^{'\mu}=S\gamma^{'\mu}S^{-1}$.

1.4 Specific representations of the γ^μ matrices

Recall $H=(-c\alpha(i\hbar)\nabla+\beta\,mc^2)$. In the non-relativistic limit, mc^2 term dominates the total energy, so it's convenient to represent $\beta=\gamma^0$ by a diagonal matrix.

Recall $Tr\beta=0$ and $\beta^2=I$, so we choose

$$\beta=\begin{pmatrix}I&0\\0&I\end{pmatrix}, \text{ where } I=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad (28)$$

(28)

α^k's anti-commutate with β and are hermitian,

$$\alpha^k=\begin{pmatrix}0&\mathrm{A}^k\\(\mathrm{A}^k)^*&0\end{pmatrix}, \qquad (29)$$

(29)

$\mathrm{A}^k: 2\times2$ matrices, anti-commutate with each other.

These properties are satisfied by the Pauli matrices, so we have

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{30}$$

from these we obtain

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{31}$$

This is the Pauli-Dirac representation of the γ^μ matrices. It's most useful for system with small kinetic energy, e.g., atomic physics.

2. Spinor Formulations of Dirac Equations

Let's consider the simplest possible problem:

Free particle at rest. ψ is a 4-component wave-function with each component satisfying the Klein-Gordon equation,

$$\psi = \chi\, e^{\frac{i}{\hbar}(p.x - Et)}, \tag{32}$$

where χ is a 4-compnent spinor and $E^2 = p^2c^2 + m^2c^4$.

Free particle at rest: $p = 0, \psi$ is independent of x,

$$H\psi = \left(-i\hbar c\alpha.\nabla + mc^2\gamma^0\right)\psi = mc^2\gamma^0\psi = E\psi. \tag{33}$$

In Pauli-Dirac representation, $\gamma^0 = diag\,(1,1,-1,-1)$, the 4-fundamental solutions are

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad E = mc^2, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E = mc^2,$$

$$\chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad E = -mc^2, \quad \chi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad E = -mc^2.$$

As we shall see, Dirac wave function describes a particle pf spin$-1/2$. χ_1, χ_2 represent spin-up and spin-down respectively with $E = mc^2$. χ_3, χ_4 represent spin-up and spin-down respectively with $E = -mc^2$. As in Klein-Gordon equation, we have negative energy solutions and they can not be discarded.

For ultra-relativistic problems, the Weyl representation is more convenient.

$$\psi_{PD} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{34}$$

In terms of ψ_A and ψ_B, the Dirac equation is

$$i\frac{\partial}{\partial x^0}\psi_A + i\sigma.\nabla\psi_B = \frac{mc}{\hbar}\psi_A,$$
$$-i\frac{\partial}{\partial x^0}\psi_B - i\sigma.\nabla\psi_A = \frac{mc}{\hbar}\psi_B. \tag{35}$$

Let's define

$$\psi_A = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2), \quad \psi_B = \frac{1}{\sqrt{2}}(\phi_2 - \phi_1) \tag{36}$$

and rewrite the Dirac equation in terms of ϕ_1 and ϕ_2,

$$i\frac{\partial}{\partial x^0}\phi_1 - i\sigma.\nabla\phi_1 = \frac{mc}{\hbar}\phi_2,$$
$$i\frac{\partial}{\partial x^0}\phi_2 + i\sigma.\nabla\phi_2 = \frac{mc}{\hbar}\phi_1. \tag{37}$$

On can see that ϕ_1 and ϕ_2 are coupled only via the mass term. In ultra-relativistic limit (or for nearly massless particle such as neutrinos), rest mass is negligible, then ϕ_1 and ϕ_2, decouple,

$$i\frac{\partial}{\partial x^0}\phi_1 - i\sigma.\nabla\phi_1 = 0,$$
$$i\frac{\partial}{\partial x^0}\phi_2 + i\sigma.\nabla\phi_2 = 0, \tag{38}$$

The 4-component wavefunction in the Weyl representation is written as

$$\psi_{Weyl} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{39}$$

Let's imaging that a massless spin $-1/2$ neutrino is described by ϕ_1, a plane wave state of a definite momentum p with energy $E=|p|c$,

$$\phi_1 \propto e^{\frac{i}{\hbar}(p.x-Et)}. \tag{40}$$

$$i\frac{\partial}{\partial x^0}\phi_1 = i\frac{1}{c}\frac{\partial}{\partial t}\phi_1 = \frac{E}{\hbar c}\phi_1,$$

$$i\sigma.\nabla\phi_1 = -\frac{1}{\hbar}\sigma.p\phi_1$$

$$\Rightarrow E\phi_1 = |p|c\phi_1 = -c\sigma.p\phi_1 \quad or \quad \frac{\sigma.p}{|p|}\phi_1 = -\phi_1. \tag{41}$$

The operator $h=\sigma.p/|p|$ is called the helicity. Physically it refers to the component of spin in the direction of motion. ϕ_1 describes a neutrino with helicity -1 (left-handed). Similarly,

$$\frac{\sigma.p}{|p|}\phi_2 = -\phi_2, \quad (h=+1, right-handed). \tag{42}$$

The $\gamma^{\mu}{}'s$ in the Weyl representation are

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

3. Spin Complex

The spin complex is perhaps the most subtle and interesting of the classical elliptic complexes. The deepest insight into its mathematical stricture can be achieved using Clifford algebra bundles [7]. Clifford also provide a unified context for treating all four of the classical elliptic complexes. In fact, one may use the Clifford algebra approach to show that the spin complex is interpretable as the square-root of plus or minus the de Rham complex. Here we shall give a more mundane treatment of the spin complex.

3.1 Notation

We begin by restricting ourselves to a four-dimensional Euclidean-signature Riemannian spin manifold M. We choose Dirac matrices obeying

$$\{\gamma^a,\gamma^b\}=\gamma^a\gamma^b+\gamma^b\gamma^a=-2\delta_{ab} \tag{43}$$

and take the representation

$$\gamma^a=\begin{pmatrix}0 & i\alpha_a\\ -i\overline{\alpha}_a & 0\end{pmatrix},\ \alpha_a=(I,-i\lambda),\ \overline{\alpha}_a=(I,i\lambda) \tag{44}$$

where $\{\lambda\}$ are the 2×2 Pauli matrices

$$\lambda_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \lambda_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \lambda_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{45}$$

Then the chiral operator γ_5 is diagonal,

$$\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{46}$$

and we may split the space of Dirac spinors $\{\psi_\alpha\}$ into two eigen-spaces of chirality ± 1:

$$\gamma_5\psi_\pm=\pm\psi_\pm \tag{47}$$

the Dirac operator D is defined using the covariant derivative with respect to the basis of

$$D=\gamma^a E_a^\mu(x)D_\mu(x)$$
$$=\gamma^a E_a^\mu(x)\left(\frac{\partial}{\partial x^\mu}+\tfrac{1}{4}[\gamma_b,\gamma_c]\omega_\mu^{ab}(x)\right), \tag{48}$$

where E_a^μ is an inverse vierbein of the metric on M and $\omega_\mu^{ab}dx^\mu$ is the spin connection $1-$forms.

It is easy to show that (Weitzenbock formula):

$$D^*D=DD^*=-g^{\mu\nu}D_\mu D_\nu+\tfrac{1}{16}[\gamma_a,\gamma_b][\gamma_c,\gamma_d]R^{abcd}$$
$$=-g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}+\dots, \tag{49}$$

so the leading part of the operator is elliptic for metrics with Euclidean signature.

Clearly the spinors $\psi_+(x)$ upon which D acts are the analogs of C^∞ sections of the fibers of the bundles, we treated in previous examples. We therefore must introduce a pair of corresponding spin bundles $\Delta_{\pm}$ over M with local coordinates $\Delta_{\pm} : (x^{\mu}, \psi_{\pm})$.

Thus we finally arrive at the following definition of the spin complex

$$D : C^{\infty}(\Delta_{+}) \to C^{\aleph}(\Delta_{-})$$
$$D^{*} : C^{\infty}(\Delta_{-}) \to C^{\infty}(\Delta_{+}). \qquad (50)$$

4. Index of the spin complex

4.1 $\hat{A}$ Genus

The $\hat{A}$ genus is the genus associated to the characteristic power series:

$$Q(z) = \frac{\sqrt{z}/2}{\sinh\left(\sqrt{z}/2\right)} = 1 - z/24 + 7z^2/5760 - .... \qquad (51)$$

The $\hat{A}$ genus of a spin manifold M is an integer, and an even integer if the dimension is 4 mod 8 (which in dimension 4 implies Rochlin's theorem), for general manifolds, the $\hat{A}$ genus is not always an integer.

The $\hat{A}$ genus is given by:

$$\hat{A}(M) = \prod_{i=1}^{n/2} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24} p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots \qquad (52)$$

4.2 Index of the spin complex

The analytic index of spin complex $(\Delta_{\pm}, D)$ is

$$Index(\Delta_{\pm}, D) = \dim Ker\, D - \dim Ker\, D^{*}$$
$$= n_{+} - n_{-}, \qquad (53)$$

where $n_{\pm}$ are the numbers of chirality ± 1 normalizable zero-frequency Dirac spinors.

The index itself is the difference between the numbers of ± 1 chirality spinors in the kernel of Dirac operator.

Now we turn to the topological index of spin complex , it can be expressed as:

$$Index(\Delta^{\pm}, D) = \int \prod_{j=1}^{l} \frac{x_j/2}{\sinh(x_j/2)} = \int_M \hat{A}(M). \qquad (54)$$

Where $n = \dim M$ is a multiple of 4. for $n = 4$, we find

$$n_{+} - n_{-} = -\frac{1}{24} P_1 \equiv -\frac{1}{24} \int_M p_1(T(M)) = +\frac{1}{24.8\pi^2} \int_M Tr(R \wedge R).$$

Hence P_1 is a multiple of 24 for any compact 4-dimensional spin manifold without boundary.

5. Conclusions

The significance of this study is that the global geometrical interpretation of Dirac equations provides the existence of solutions of the Dirac equations. In fact the topological invariants introduced are the key points that affect the relation between the analytical techniques and the topological and geometrical data.

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